

## RESEARCH ARTICLE

Gain-scheduled  $H_\infty$  Control via  
Parameter-Dependent Lyapunov FunctionsSunan Chumalee<sup>a</sup> and James F Whidborne<sup>b\*</sup><sup>a</sup>*Research and Development Centre for Space and Aeronautical Science and Technology,  
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Synthesizing a gain-scheduled output feedback  $H_\infty$  controller via parameter-dependent Lyapunov functions for linear parameter-varying (LPV) plant models involves solving an infinite number of linear matrix inequalities (LMIs). In practice, for affine LPV models, a finite number of LMIs can be achieved using convexifying techniques. This paper proposes an alternative approach to achieve a finite number of LMIs. By simple manipulations on the bounded real lemma inequality, a symmetric matrix polytope inequality can be formed. Hence, the LMIs need only to be evaluated at all vertices of such a symmetric matrix polytope. In addition, a construction technique of the intermediate controller variables is also proposed as an affine matrix-valued function in the polytopic coordinates of the scheduled parameters. Computational results on a numerical example using the approach were compared with those from a multi-convexity approach in order to demonstrate the impacts of the approach on parameter-dependent Lyapunov-based stability and performance analysis. Furthermore, numerical simulation results show the effectiveness of these proposed techniques.

**Keywords:** Gain-scheduling control, parameter-dependent Lyapunov functions, linear parameter varying (LPV) systems, linear matrix inequality (LMI), nonlinear control, robust control.

## 1. Introduction

The dynamic characteristics of nonlinear plants vary following their operating conditions. Conventional gain-scheduling techniques can be used to handle this nonlinear property. Although this design approach has been successfully and popularly implemented in many engineering applications in order to cover the entire operating range of the system plants, it comes with no guarantees on the robustness, performance, or even nominal stability of the overall gain scheduled design (Shamma and Athans 1990). An advanced robust gain-scheduling technique, namely linear parameter-varying (LPV) control (Shamma and Athans 1991), provides an alternative approach that is often used to handle plant uncertainty and nonlinearity. Importantly, an LPV controller theoretically guarantees stability, robustness, and performance properties of the closed-loop system (Wu 2001, Apkarian and Adams 1998, Becker and Packard 1994).

An LPV plant model was first introduced by Shamma and Athans (1991) whereby its dynamic characteristics vary, following some time-varying parameters whose values are unknown *a priori* but can be measured in real-time and lie in some set bounded by known minimum and maximum possible values. Algebraic manipulation methods such as Jacobian linearization (Ostergaard

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et al. 2009, Corno et al. 2008, Lu et al. 2006), function substitution (Shin et al. 2002), or state transformation (Balas 2002, Shamma and Cloutier 1993), are normally used to derive an LPV model from the original nonlinear model. Moreover, in the literature, there are several different varieties of LPV models, e.g. grid LPV model (Corno et al. 2008, Lu et al. 2006, Wu et al. 2002, 1996), affine LPV model (Apkarian and Adams 1998, Apkarian 1997, Apkarian et al. 1995) (or polytopic LPV model), tensor-product (TP) convex polytopic model (Baranyi et al. 2006, Baranyi 2005, 2004), etc.; these have been introduced for gain-scheduled  $H_\infty$  control synthesis and analysis which is usually based on single quadratic Lyapunov function (Apkarian et al. 1995, Becker and Packard 1994) or parameter-dependent Lyapunov function, e.g. parameter-dependent (Corno et al. 2008, Wu et al. 2002, Apkarian and Adams 1998, Wu et al. 1996), affine parameter-dependent (Gahinet et al. 1996), piecewise-affine parameter-dependent (Lim and How 2003, 2002), blending parameter-dependent (Shin et al. 2002), multiple parameter-dependent Lyapunov functions (Lu et al. 2006, Lu and Wu 2004).

Using a single quadratic Lyapunov function, for both the affine LPV model (Apkarian et al. 1995) and the TP convex polytopic model (Chumalee and Whidborne 2009) cases, a finite number of LMIs need only to be evaluated at all vertices while, for the grid LPV models (Wu et al. 1995) case, an infinite number of LMIs have to be evaluated at all points over the entire parameter space in order to determine a pair of positive definite symmetric matrices  $(\mathbf{X}, \mathbf{Y})$ . However, in practice, the symmetric matrices  $(\mathbf{X}, \mathbf{Y})$  can be determined from a finite number of LMIs by gridding the entire parameter space with a non-dense set of grid points. Having determined the symmetric matrices  $(\mathbf{X}, \mathbf{Y})$ , a more dense grid points set can be tested with these determined symmetric matrices  $(\mathbf{X}, \mathbf{Y})$  to check whether the LMIs are still satisfied (Wu et al. 2002, 1996). If not, this process is repeated with a denser grid until the symmetric matrices  $(\mathbf{X}, \mathbf{Y})$ , that satisfy the LMIs for all points over the entire parameters space, are obtained (Wu et al. 2002, 1996, Lim and How 2003). However the result of such a heuristic gridding technique is not necessarily reliable and the analysis result is dependent on the choice of gridding points (Wang and Balakrishnan 2002). In addition, for a grid LPV model case, the resulting gain-scheduled controller has high computational on-line complexity at the gain-scheduling level (Wu et al. 1995) while, for the other two cases, the gain-scheduled controller is constructed as an affine matrix-valued function in the polytopic coordinates of the scheduled parameters (Apkarian et al. 1995, Chumalee and Whidborne 2009).

In general, the single quadratic Lyapunov function is more conservative than the parameter-dependent Lyapunov function when the parameters are time-invariant or slowly varying (Gahinet et al. 1996). In addition, when the parameters have a large variation, the piecewise-affine parameter-dependent (Lim and How 2003, 2002), blending parameter-dependent (Shin et al. 2002), and multiple parameter-dependent Lyapunov functions (Lu et al. 2006, Lu and Wu 2004) are less conservative than the parameter-dependent Lyapunov functions. This result from the fact that an LPV model with a large parameter variation can be modelled as a switching linear parameter-varying (SLPV) system which can be made discontinuous along the switching surface by dividing the entire parameters spaces into parameters subsets that are small variation regions. Solving LMIs with the parameters subsets improve the performance measure ( $\gamma$ ). Moreover, Yan and Ozbay (2007) provide sufficient conditions to guarantee the stability of the SLPV systems in terms of the dwell time and the average dwell time, where a switching signal of the SLPV systems with dwell time  $\tau$  means a time interval between any two consecutive switching from one model to another model is equal to or greater than  $\tau$ . However, using parameter-dependent (Corno et al. 2008, Wu et al. 2002, Apkarian and Adams 1998, Wu et al. 1996), blending parameter-dependent (Shin et al. 2002), and multiple parameter-dependent Lyapunov functions (Lu et al. 2006, Lu and Wu 2004), an infinite number of LMIs have to be evaluated at all points over the entire space of parameters. Furthermore, the resulting gain-scheduled controller requires more complex on-line computations at the gain-scheduling level (Corno et al. 2008, Lu et al. 2006, Lu and Wu 2004, Shin et al. 2002, Wu et al. 2002, Apkarian and Adams 1998).

In this paper, we consider the design problem of gain-scheduled controllers with guaranteed  $L_2$ -gain performance for a class of affine LPV systems using parameter-dependent Lyapunov functions. By simple manipulations on the parameter-dependent Lyapunov functions or the bounded real lemma (Apkarian et al. 1995) inequality, a symmetric matrix polytope form of these inequalities is obtained. Hence, the LMIs need only to be evaluated at all vertices of the system state-space model matrices and the variation rate of the scheduled parameters. In addition, the intermediate controller variables, i.e.  $\hat{\mathbf{A}}_k(\boldsymbol{\theta})$ ,  $\hat{\mathbf{B}}_k(\boldsymbol{\theta})$ ,  $\hat{\mathbf{C}}_k(\boldsymbol{\theta})$  and  $\mathbf{D}_k(\boldsymbol{\theta})$ , are proposed to be constructed as an affine matrix-valued function in the polytopic coordinates of the scheduled parameters, hence this reduces the computational burden and eases controller implementation. Furthermore, they are applicable to both regular and singular problems without the need for constraints on the  $\mathbf{D}_{12}$  and  $\mathbf{D}_{21}$  matrices of LPV models. The performance of the proposed approach is tested on a numerical example (Leith and Leithead 1999) that is known to cause difficulties for LPV controllers.

The organization of the paper is as follows. A vertex-type stability analysis technique for affine LPV systems based on parameter-dependent Lyapunov functions is briefly summarized in the next section. In section 3, gain-scheduled output feedback  $H_\infty$  control synthesis methods for affine LPV systems using parameter-dependent Lyapunov functions are proposed. Numerical comparisons of LPV synthesis techniques using multi-convexity (Apkarian and Tuan 2000) and the proposed method are presented in section 4, where numerical simulation results are also presented. This paper concludes with some comments.

The notation used in the paper is standard.  $\mathbb{R}^{m \times n}$  is the set of real  $m \times n$  matrices. The identity matrix size  $p \times p$  and zero matrix size  $m \times n$  are denoted  $\mathbf{I}_p$  and  $\mathbf{0}_{m \times n}$ , respectively. The transpose, inverse, and Moore-Penrose pseudo-inverse of a matrix  $\mathbf{M}$  are denoted  $\mathbf{M}^T$ ,  $\mathbf{M}^{-1}$ , and  $\mathbf{M}^+$ , respectively.  $\Theta \times \Phi$  denotes the Cartesian product of two sets  $\Theta$  and  $\Phi$ . For real symmetric matrices  $\mathbf{M}$ , the notation  $\mathbf{M} < 0$  stands for negative definite and means that all the eigenvalues of  $\mathbf{M}$  are negative, that is,  $\mathbf{x}^T \mathbf{M} \mathbf{x} < 0$  for all nonzero vectors  $\mathbf{x}$ . The notation  $\dagger$  denotes matrices without interest. In large symmetric matrix expressions, the notation  $\star$  represents a symmetric matrix block. For instance,

$$\mathbf{X}(\boldsymbol{\theta})\mathbf{A}(\boldsymbol{\theta}) + \hat{\mathbf{B}}_k(\boldsymbol{\theta})\mathbf{C}_2 + (\star) = \mathbf{X}(\boldsymbol{\theta})\mathbf{A}(\boldsymbol{\theta}) + \hat{\mathbf{B}}_k(\boldsymbol{\theta})\mathbf{C}_2 + \mathbf{A}^T(\boldsymbol{\theta})\mathbf{X}(\boldsymbol{\theta}) + \mathbf{C}_2^T \hat{\mathbf{B}}_k^T(\boldsymbol{\theta})$$

## 2. Stability Analysis using Parameter-Dependent Lyapunov Functions

Following Gahinet et al. (1996), an affine LPV system is given by,

$$\dot{\mathbf{x}}(t) = \mathbf{A}(\boldsymbol{\theta}(t))\mathbf{x}(t), \quad \mathbf{x}(0) = \mathbf{x}_0 \quad (1)$$

where  $t \in \mathbb{R}$  is time,  $\mathbf{x} \in \mathbb{R}^p$  is the state vector,  $\boldsymbol{\theta}(t) = [\theta_1(t), \dots, \theta_n(t)]^T \in \mathbb{R}^n$  is a vector of time-varying real parameters which is assumed to be measured in real-time and  $n$  is the total number of time-varying parameters. The plant state matrix  $\mathbf{A}(\boldsymbol{\theta}(t))$  is a continuous mapping matrix functions  $\mathbf{A} : \mathbb{R}^n \rightarrow \mathbb{R}^{p \times p}$  and is assumed to depend affinely on the vector parameters  $\boldsymbol{\theta}(t)$ . That is

$$\mathbf{A}(\boldsymbol{\theta}(t)) = \mathbf{A}_0 + \theta_1(t)\mathbf{A}_1 + \dots + \theta_n(t)\mathbf{A}_n \quad (2)$$

where  $\mathbf{A}_0, \mathbf{A}_1, \dots, \mathbf{A}_n$  are known fixed matrices. We also assume that each parameter  $\theta_i(t)$  lies between known extremal values  $\underline{\theta}_i$  and  $\bar{\theta}_i$ ,  $\theta_i(t) \in [\underline{\theta}_i, \bar{\theta}_i]$ , and  $\boldsymbol{\theta}(t)$  lies in a polytope  $\Theta$ ,  $\boldsymbol{\theta}(t) \in \Theta$ . The rate of variation  $\dot{\theta}_i(t)$  is well defined at all times and satisfies  $\dot{\theta}_i(t) \in [\underline{v}_i, \bar{v}_i]$  and  $\dot{\boldsymbol{\theta}}(t)$  lies in a polytope  $\Phi$ ,  $\dot{\boldsymbol{\theta}}(t) \in \Phi$ .

A matrix polytope is defined as the convex hull of a finite number of matrix vertices  $\mathbf{N}_i$  with the same dimensions (Apkarian et al. 1995). That is,

$$Co\{\mathbf{N}_1, \mathbf{N}_2, \dots, \mathbf{N}_r\} := \left\{ \sum_{i=1}^r \alpha_i \mathbf{N}_i : \alpha_i \geq 0, \sum_{i=1}^r \alpha_i = 1 \right\} \quad (3)$$

The plant state matrix  $\mathbf{A}(\boldsymbol{\theta}(t))$  can also be written as a convex combination of the matrix vertices as

$$\mathbf{A}(\boldsymbol{\theta}(t)) = Co\{\hat{\mathbf{A}}_1, \hat{\mathbf{A}}_2, \dots, \hat{\mathbf{A}}_m\} = \alpha_1 \hat{\mathbf{A}}_1 + \alpha_2 \hat{\mathbf{A}}_2 + \dots + \alpha_m \hat{\mathbf{A}}_m \quad (4)$$

where  $m = 2^n$ ,

$$\begin{bmatrix} \hat{\mathbf{A}}_1 \\ \hat{\mathbf{A}}_2 \\ \hat{\mathbf{A}}_3 \\ \vdots \\ \hat{\mathbf{A}}_m \end{bmatrix} = \begin{bmatrix} 1 & \underline{\theta}_1 & \underline{\theta}_2 & \dots & \underline{\theta}_{n-1} & \underline{\theta}_n \\ 1 & \underline{\theta}_1 & \underline{\theta}_2 & \dots & \underline{\theta}_{n-1} & \underline{\theta}_n \\ 1 & \underline{\theta}_1 & \underline{\theta}_2 & \dots & \underline{\theta}_{n-1} & \underline{\theta}_n \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \bar{\theta}_1 & \bar{\theta}_2 & \dots & \bar{\theta}_{n-1} & \bar{\theta}_n \end{bmatrix} \begin{bmatrix} \mathbf{A}_0 \\ \mathbf{A}_1 \\ \vdots \\ \mathbf{A}_n \end{bmatrix} \quad (5)$$

and, following Pellanda et al. (2002, Algorithm 3.1), in order to compute  $\alpha_i$ , we first compute the normalized co-ordinates

$$\alpha_{\theta_j} = \frac{\bar{\theta}_j - \theta_j(t)}{\bar{\theta}_j - \underline{\theta}_j}, \quad j = 1, \dots, n \quad (6)$$

Then, for each vertex  $\boldsymbol{\Theta}_i$ ,  $i = 1, \dots, m$ , the corresponding polytopic co-ordinates are calculated by

$$\alpha_i = \prod_{j=1}^n \tilde{\alpha}_{\theta_j}, \quad \tilde{\alpha}_{\theta_j} = \begin{cases} \alpha_{\theta_j}, & \text{if } \underline{\theta}_j \text{ is a co-ordinate of } \boldsymbol{\Theta}_i; \\ 1 - \alpha_{\theta_j}, & \text{if } \bar{\theta}_j \text{ is a co-ordinate of } \boldsymbol{\Theta}_i. \end{cases} \quad (7)$$

**Lemma 2.1:** (Chumalee 2010, Lemma 3.3.1.) Given a symmetric matrix polytope,  $\mathbf{M}(\boldsymbol{\theta}(t)) \in \mathbb{R}^{p \times p}$ , for which  $\mathbf{M}(\boldsymbol{\theta}(t)) = \sum_{i=1}^m \alpha_i \hat{\mathbf{M}}_i$ , where  $\alpha_i$  is determined using (6) and (7), is a negative definite symmetric matrix for all possible parameter trajectories,  $\mathbf{M}(\boldsymbol{\theta}(t)) < 0$ ,  $\forall \boldsymbol{\theta}(t) \in \boldsymbol{\Theta}$ , if and only if  $\hat{\mathbf{M}}_i < 0$ ,  $i = 1, \dots, m$ .

**Proof:** Sufficiency: since  $\alpha_i \in [0, 1]$  for  $i = 1, \dots, m$  and  $\sum_{i=1}^m \alpha_i = 1$  then there is always at least one  $i$  such that  $\alpha_i > 0$ . Thus  $\hat{\mathbf{M}}_i < 0$  for all  $i$  implies  $\sum_{i=1}^m \alpha_i \hat{\mathbf{M}}_i < 0$  and hence  $\mathbf{M}(\boldsymbol{\theta}(t)) < 0$  for all  $\boldsymbol{\theta}(t) \in \boldsymbol{\Theta}$ .

Necessity: from (6) and (7), for all  $j$  there exists a  $\boldsymbol{\theta}(t) \in \boldsymbol{\Theta}$  such that there is an  $\alpha_j = 1$  and  $\alpha_i = 0$  for  $i = 1, \dots, m$ ,  $i \neq j$ . Hence for all  $j$  there exists a  $\boldsymbol{\theta}(t) \in \boldsymbol{\Theta}$  such that  $\mathbf{M}(\boldsymbol{\theta}(t)) = \hat{\mathbf{M}}_j$  and so it is necessary that  $\hat{\mathbf{M}}_j < 0$  for all  $j$ .  $\square$

For convenience, in the following sections, we will henceforth often drop the dependence of  $\boldsymbol{\theta}$  on  $t$ . Following Gahinet et al. (1996), the system (1) is said to be parameter-dependent stable if there exists a continuously differentiable parameter-dependent Lyapunov function  $V(\mathbf{x}, \boldsymbol{\theta}) = \mathbf{x}^T \mathbf{P}(\boldsymbol{\theta}) \mathbf{x}$  whose derivative,  $\dot{V}(\mathbf{x}, \boldsymbol{\theta})$ , is negative along all state trajectories and is given by  $\dot{V}(\mathbf{x}, \boldsymbol{\theta}) = \mathbf{x}^T (\mathbf{A}^T(\boldsymbol{\theta}) \mathbf{P}(\boldsymbol{\theta}) + \mathbf{P}(\boldsymbol{\theta}) \mathbf{A}(\boldsymbol{\theta}) + \dot{\mathbf{P}}(\boldsymbol{\theta})) \mathbf{x}$ . This is equivalent to the existence of a  $\mathbf{P}(\boldsymbol{\theta}) = \mathbf{P}^T(\boldsymbol{\theta})$

such that (Gahinet et al. 1996)

$$\begin{aligned} \mathbf{P}(\boldsymbol{\theta}) &> 0, \\ \mathbf{A}^T(\boldsymbol{\theta})\mathbf{P}(\boldsymbol{\theta}) + \mathbf{P}(\boldsymbol{\theta})\mathbf{A}(\boldsymbol{\theta}) + \dot{\mathbf{P}}(\boldsymbol{\theta}) &< 0, \quad \forall(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) \in \boldsymbol{\Theta} \times \boldsymbol{\Phi} \end{aligned} \quad (8)$$

Although an exact parameter-dependent function for a continuously differentiable parameter-dependent Lyapunov variable  $\mathbf{P}(\boldsymbol{\theta})$  is still not established, a basis parameter-dependent function for the parameter-dependent Lyapunov variable is suggested in Wu et al. (2002), Apkarian and Adams (1998), Wu et al. (1996) and is to copy the plant's parameter-dependent function. Therefore, we can constrain the basis parameter-dependent function for the parameter-dependent Lyapunov variable to vary in an affine fashion

$$\mathbf{P}(\boldsymbol{\theta}) = \mathbf{P}_0 + \theta_1 \mathbf{P}_1 + \cdots + \theta_n \mathbf{P}_n = \alpha_1 \hat{\mathbf{P}}_1 + \alpha_2 \hat{\mathbf{P}}_2 + \cdots + \alpha_m \hat{\mathbf{P}}_m \quad (9)$$

where  $m = 2^n$ ,  $\alpha_i$  is determined using (6) and (7) and

$$\begin{bmatrix} \hat{\mathbf{P}}_1 \\ \hat{\mathbf{P}}_2 \\ \hat{\mathbf{P}}_3 \\ \vdots \\ \hat{\mathbf{P}}_m \end{bmatrix} = \begin{bmatrix} 1 & \underline{\theta}_1 & \underline{\theta}_2 & \cdots & \underline{\theta}_{n-1} & \underline{\theta}_n \\ 1 & \underline{\theta}_1 & \underline{\theta}_2 & \cdots & \underline{\theta}_{n-1} & \underline{\theta}_n \\ 1 & \underline{\theta}_1 & \underline{\theta}_2 & \cdots & \underline{\theta}_{n-1} & \underline{\theta}_n \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \bar{\theta}_1 & \bar{\theta}_2 & \cdots & \bar{\theta}_{n-1} & \bar{\theta}_n \end{bmatrix} \begin{bmatrix} \mathbf{P}_0 \\ \mathbf{P}_1 \\ \vdots \\ \mathbf{P}_n \end{bmatrix} \quad (10)$$

Differentiating (9) with respect to time gives

$$\dot{\mathbf{P}}(\boldsymbol{\theta}) = \dot{\theta}_1 \mathbf{P}_1 + \cdots + \dot{\theta}_n \mathbf{P}_n = \beta_1 \tilde{\mathbf{P}}_1 + \beta_2 \tilde{\mathbf{P}}_2 + \cdots + \beta_m \tilde{\mathbf{P}}_m \quad (11)$$

where  $\beta_i$  can be determined in a similar manner to  $\alpha_i$  using (6) and (7) and

$$\begin{bmatrix} \tilde{\mathbf{P}}_1 \\ \tilde{\mathbf{P}}_2 \\ \tilde{\mathbf{P}}_3 \\ \vdots \\ \tilde{\mathbf{P}}_m \end{bmatrix} = \begin{bmatrix} 0 & \underline{v}_1 & \underline{v}_2 & \cdots & \underline{v}_{n-1} & \underline{v}_n \\ 0 & \underline{v}_1 & \underline{v}_2 & \cdots & \underline{v}_{n-1} & \underline{v}_n \\ 0 & \underline{v}_1 & \underline{v}_2 & \cdots & \underline{v}_{n-1} & \underline{v}_n \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \bar{v}_1 & \bar{v}_2 & \cdots & \bar{v}_{n-1} & \bar{v}_n \end{bmatrix} \begin{bmatrix} \mathbf{P}_0 \\ \mathbf{P}_1 \\ \vdots \\ \mathbf{P}_n \end{bmatrix} \quad (12)$$

Substituting (4), (9) and (11) into (8), and recalling that  $\sum_{i=1}^m \alpha_i = 1$  and  $\sum_{i=1}^m \beta_i = 1$ , we get

$$\begin{aligned} \sum_{i=1}^m \alpha_i \hat{\mathbf{P}}_i &> 0, \quad \sum_{i=1}^m \sum_{k=1}^m \alpha_i^2 \beta_k \left( \hat{\mathbf{A}}_i^T \hat{\mathbf{P}}_i + \hat{\mathbf{P}}_i \hat{\mathbf{A}}_i + \tilde{\mathbf{P}}_k \right) \\ + 2 \sum_{i=1}^{m-1} \sum_{j=i+1}^m \sum_{k=1}^m \alpha_i \alpha_j \beta_k &\left( \frac{1}{2} \left( \hat{\mathbf{A}}_i^T \hat{\mathbf{P}}_j + \hat{\mathbf{P}}_j \hat{\mathbf{A}}_i + \hat{\mathbf{A}}_j^T \hat{\mathbf{P}}_i + \hat{\mathbf{P}}_i \hat{\mathbf{A}}_j + 2\tilde{\mathbf{P}}_k \right) \right) < 0, \end{aligned} \quad (13)$$

$\forall(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) \in \boldsymbol{\Theta} \times \boldsymbol{\Phi}$ . As  $\alpha_i^2 \beta_k \in [0, 1]$ ,  $i, k = 1, \dots, m$ ,  $2\alpha_i \alpha_j \beta_k \in [0, 0.5]$ ,  $i = 1, \dots, m-1$ ,  $j = i+1, \dots, m$ ,  $k = 1, \dots, m$ , and  $\sum_{i=1}^m \sum_{k=1}^m \alpha_i^2 \beta_k + 2 \sum_{i=1}^{m-1} \sum_{j=i+1}^m \sum_{k=1}^m \alpha_i \alpha_j \beta_k = 1$ , by Lemma 2.1 solving (13) for parameter-dependent Lyapunov variable  $\mathbf{P}(\boldsymbol{\theta}) = \sum_{i=1}^m \alpha_i \hat{\mathbf{P}}_i$  need only to be done at all vertices. Hence we get the following proposition.

**Proposition 2.2:** *The system (1) is parameter-dependent stable whenever there exist a positive definite symmetric matrix  $\hat{\mathbf{P}}_i$ ,  $i = 1, 2, \dots, m$ , such that the following LMI conditions hold*

$$\hat{\mathbf{P}}_i > 0, \quad (14)$$

$$\hat{\mathbf{A}}_i^T \hat{\mathbf{P}}_i + \hat{\mathbf{P}}_i \hat{\mathbf{A}}_i + \tilde{\mathbf{P}}_k < 0, \quad (15)$$

$$\hat{\mathbf{A}}_i^T \hat{\mathbf{P}}_j + \hat{\mathbf{P}}_j \hat{\mathbf{A}}_i + \hat{\mathbf{A}}_j^T \hat{\mathbf{P}}_i + \hat{\mathbf{P}}_i \hat{\mathbf{A}}_j + 2\tilde{\mathbf{P}}_k < 0, \quad (16)$$

for  $i, k = 1, \dots, m$  and  $1 \leq i < j \leq m$

Note that  $\hat{\mathbf{P}}_i$  and  $\tilde{\mathbf{P}}_i$ ,  $i = 1, \dots, m$  map to  $\mathbf{P}_j$ ,  $j = 1, \dots, n$  using (10) and (12), respectively. In addition, the numbers of LMIs for (14)-(16) are  $m$ ,  $m^2$  and  $m^2(m-1)/2$ , respectively. Therefore, the total number of LMIs to be solved is  $m(m^2 + m + 2)/2$ .

Some similar results to Proposition 2.2 appear in Jadbabaie (1999), and also in Bernal and Guerra (2010), Rhee and Won (2006) and Tanaka et al. (2003), but in the context of fuzzy logic control systems. Note also that there are other approaches that could be used to obtain a finite number of LMIs, e.g. multi-convexity (Gahinet et al. 1996, Apkarian and Tuan 2000), S-procedure (Feron et al. 1995), gridding parameter space (Corno et al. 2008, Lu et al. 2006, Wu et al. 2002, 1996), etc.

**Proposition 2.3:** *(Apkarian and Tuan 2000, Multi-convexity approach) Assume that  $\dot{\boldsymbol{\theta}} = 0$ , The system (1) is parameter-dependent stable whenever there exist a positive definite symmetric matrix  $\hat{\mathbf{P}}_i$ ,  $i = 1, 2, \dots, m$ , and scalars  $\lambda_i$ ,  $i = 1, 2, \dots, m$ , such that the following LMI conditions hold*

$$\hat{\mathbf{P}}_k > 0, \quad (17)$$

$$\lambda_k \geq 0, \quad (18)$$

$$\hat{\mathbf{A}}_k^T \hat{\mathbf{P}}_k + \hat{\mathbf{P}}_k \hat{\mathbf{A}}_k < -\lambda_k \mathbf{I}, \quad (19)$$

$$\hat{\mathbf{A}}_i^T \hat{\mathbf{P}}_i + \hat{\mathbf{P}}_i \hat{\mathbf{A}}_i + \hat{\mathbf{A}}_j^T \hat{\mathbf{P}}_j + \hat{\mathbf{P}}_j \hat{\mathbf{A}}_j - (\hat{\mathbf{A}}_i^T \hat{\mathbf{P}}_j + \hat{\mathbf{P}}_j \hat{\mathbf{A}}_i + \hat{\mathbf{A}}_j^T \hat{\mathbf{P}}_i + \hat{\mathbf{P}}_i \hat{\mathbf{A}}_j) \geq -(\lambda_i + \lambda_j) \mathbf{I}, \quad (20)$$

for  $k = 1, \dots, m$  and  $1 \leq i < j \leq m$

**Corollary 2.4:** *Assume that  $\dot{\boldsymbol{\theta}} = 0$ . If Proposition 2.3 is satisfied then Proposition 2.2 is satisfied.*

**Proof:** First, we show that if the inequality (19) is satisfied then the inequality (15) is satisfied. Let  $\hat{\mathbf{A}}_k^T \hat{\mathbf{P}}_k + \hat{\mathbf{P}}_k \hat{\mathbf{A}}_k = \mathbf{M}_k$  and  $\mathbf{M}_k + \lambda_k \mathbf{I} < 0$  hence, for all nonzero vector  $\mathbf{x} \in \mathbb{R}^p$ ,  $\mathbf{x}^T \mathbf{M}_k \mathbf{x} + \lambda_k \mathbf{x}^T \mathbf{x} < 0$  if and only if  $\mathbf{x}^T \mathbf{M}_k \mathbf{x} < -\lambda_k \mathbf{x}^T \mathbf{x} \leq 0$  since  $\lambda_k \geq 0$  and  $\mathbf{x}^T \mathbf{x} > 0$ . This yields  $\mathbf{x}^T \mathbf{M}_k \mathbf{x} < 0$  or  $\hat{\mathbf{A}}_k^T \hat{\mathbf{P}}_k + \hat{\mathbf{P}}_k \hat{\mathbf{A}}_k < 0$ .

Next, we show that if the inequality (20) is satisfied then the inequality (16) is satisfied. Let  $\hat{\mathbf{A}}_i^T \hat{\mathbf{P}}_i + \hat{\mathbf{P}}_i \hat{\mathbf{A}}_i + \hat{\mathbf{A}}_j^T \hat{\mathbf{P}}_j + \hat{\mathbf{P}}_j \hat{\mathbf{A}}_j = \mathbf{M}$  and  $\hat{\mathbf{A}}_i^T \hat{\mathbf{P}}_j + \hat{\mathbf{P}}_j \hat{\mathbf{A}}_i + \hat{\mathbf{A}}_j^T \hat{\mathbf{P}}_i + \hat{\mathbf{P}}_i \hat{\mathbf{A}}_j = \mathbf{M}_{ij}$ . We have  $\mathbf{x}^T \mathbf{M} \mathbf{x} - \mathbf{x}^T \mathbf{M}_{ij} \mathbf{x} + (\lambda_i + \lambda_j) \mathbf{x}^T \mathbf{x} \geq 0$ , for all nonzero vectors  $\mathbf{x} \in \mathbb{R}^p$ . From (19), we have  $\mathbf{x}^T \mathbf{M} \mathbf{x} + (\lambda_i + \lambda_j) \mathbf{x}^T \mathbf{x} < 0$ , therefore  $0 > \mathbf{x}^T \mathbf{M} \mathbf{x} + (\lambda_i + \lambda_j) \mathbf{x}^T \mathbf{x} \geq \mathbf{x}^T \mathbf{M}_{ij} \mathbf{x}$ . This yields  $\mathbf{x}^T \mathbf{M}_{ij} \mathbf{x} < 0$  or  $\hat{\mathbf{A}}_i^T \hat{\mathbf{P}}_j + \hat{\mathbf{P}}_j \hat{\mathbf{A}}_i + \hat{\mathbf{A}}_j^T \hat{\mathbf{P}}_i + \hat{\mathbf{P}}_i \hat{\mathbf{A}}_j < 0$   $\square$

The corollary above shows that the set of possible solutions for Proposition 2.3 is a subset of that for Proposition 2.2 which is more general since  $\hat{\mathbf{A}}_k^T \hat{\mathbf{P}}_k + \hat{\mathbf{P}}_k \hat{\mathbf{A}}_k$  and  $\hat{\mathbf{A}}_i^T \hat{\mathbf{P}}_j + \hat{\mathbf{P}}_j \hat{\mathbf{A}}_i + \hat{\mathbf{A}}_j^T \hat{\mathbf{P}}_i + \hat{\mathbf{P}}_i \hat{\mathbf{A}}_j$  can be less than a negative definite symmetric matrix rather than they are just less than a diagonal negative definite matrix  $-\lambda_k \mathbf{I}$  and  $-(\lambda_i + \lambda_j) \mathbf{I}$ , respectively. In addition, Proposition 2.2 shows that the determination of a negative definite symmetric matrix is not necessary, hence,

comparing with Proposition 2.3, the number of LMIs, decision variables and the computational time are reduced while the achieved performance  $\gamma$  level is improved.

### 3. Controller Synthesis using Parameter-Dependent Lyapunov Functions

In the previous section, a sufficient condition to guarantee the stability property of the LPV closed-loop system using parameter-dependent Lyapunov functions has been presented in which the analysis conditions can be represented in the form of a finite number of LMIs. Next, we consider the problem of designing a gain-scheduled output feedback  $H_\infty$  control with guaranteed  $L_2$ -gain performance for a class of affine LPV systems for which the proposed techniques in the previous section can be directly extended to synthesizing a gain-scheduled  $H_\infty$  controller.

Consider a generalized affine LPV model with state-space realization taken from Apkarian et al. (1995) is

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}(\boldsymbol{\theta})\mathbf{x} + \mathbf{B}_1(\boldsymbol{\theta})\mathbf{w} + \mathbf{B}_2\mathbf{u} \\ \mathbf{z} &= \mathbf{C}_1(\boldsymbol{\theta})\mathbf{x} + \mathbf{D}_{11}(\boldsymbol{\theta})\mathbf{w} + \mathbf{D}_{12}\mathbf{u} \\ \mathbf{y} &= \mathbf{C}_2\mathbf{x} + \mathbf{D}_{21}\mathbf{w}\end{aligned}\quad (21)$$

where  $\mathbf{x} \in \mathbb{R}^p$  is the state vector,  $\mathbf{w} \in \mathbb{R}^{m_1}$  is the generalized disturbance vector,  $\mathbf{u} \in \mathbb{R}^{m_2}$  is the control input vector,  $\mathbf{z} \in \mathbb{R}^{q_1}$  is the controlled variable or error vector,  $\mathbf{y} \in \mathbb{R}^{q_2}$  is the measurement output vector,  $\boldsymbol{\theta} \in \boldsymbol{\Theta}$ ,  $\dot{\boldsymbol{\theta}} \in \boldsymbol{\Phi}$ , and continuous mapping matrix functions  $\mathbf{A} : \mathbb{R}^n \rightarrow \mathbb{R}^{p \times p}$ ,  $\mathbf{B}_1 : \mathbb{R}^n \rightarrow \mathbb{R}^{p \times m_1}$ ,  $\mathbf{C}_1 : \mathbb{R}^n \rightarrow \mathbb{R}^{q_1 \times p}$  and  $\mathbf{D}_{11} : \mathbb{R}^n \rightarrow \mathbb{R}^{q_1 \times m_1}$ . Note that the  $\mathbf{A}(\cdot)$ ,  $\mathbf{B}_1(\cdot)$ ,  $\mathbf{C}_1(\cdot)$  and  $\mathbf{D}_{11}(\cdot)$  matrices can also be written as a convex combination of the matrix vertices in a similar manner to (9):

$$\begin{pmatrix} \mathbf{A}(\boldsymbol{\theta}) & \mathbf{B}_1(\boldsymbol{\theta}) & \mathbf{B}_2 \\ \mathbf{C}_1(\boldsymbol{\theta}) & \mathbf{D}_{11}(\boldsymbol{\theta}) & \mathbf{D}_{12} \\ \mathbf{C}_2 & \mathbf{D}_{21} & \mathbf{0} \end{pmatrix} = \sum_{i=1}^m \alpha_i \begin{pmatrix} \hat{\mathbf{A}}_i & \hat{\mathbf{B}}_{1_i} & \mathbf{B}_2 \\ \hat{\mathbf{C}}_{1_i} & \hat{\mathbf{D}}_{11_i} & \mathbf{D}_{12} \\ \mathbf{C}_2 & \mathbf{D}_{21} & \mathbf{0} \end{pmatrix}\quad (22)$$

The gain-scheduled output feedback  $H_\infty$  control problem using the parameter-dependent Lyapunov functions is to compute a dynamic LPV controller,  $\mathbf{K}(\boldsymbol{\theta})$ , with state-space equations

$$\begin{aligned}\dot{\mathbf{x}}_k &= \mathbf{A}_k(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}})\mathbf{x}_k + \mathbf{B}_k(\boldsymbol{\theta})\mathbf{y} \\ \mathbf{u} &= \mathbf{C}_k(\boldsymbol{\theta})\mathbf{x}_k + \mathbf{D}_k(\boldsymbol{\theta})\mathbf{y}\end{aligned}\quad (23)$$

which stabilizes the closed-loop system, (21) and (23), and minimizes the closed-loop quadratic  $H_\infty$  performance,  $\gamma$ , ensures the induced  $L_2$ -norm of the operator mapping the disturbance signal  $\mathbf{w}$  into the controlled signal  $\mathbf{z}$  is bounded by  $\gamma$

$$\int_0^{t_1} \mathbf{z}^T \mathbf{z} dt \leq \gamma^2 \int_0^{t_1} \mathbf{w}^T \mathbf{w} dt, \quad \forall t_1 \geq 0 \quad (24)$$

along all possible parameter trajectories,  $\forall(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) \in \boldsymbol{\Theta} \times \boldsymbol{\Phi}$ . The assumed dimensions of the controller matrices are  $\mathbf{A}_k : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^{p \times p}$ ,  $\mathbf{B}_k : \mathbb{R}^n \rightarrow \mathbb{R}^{p \times q_2}$ ,  $\mathbf{C}_k : \mathbb{R}^n \rightarrow \mathbb{R}^{m_2 \times p}$ , and  $\mathbf{D}_k : \mathbb{R}^n \rightarrow \mathbb{R}^{m_2 \times q_2}$ . Note that  $\mathbf{A}$  and  $\mathbf{A}_k$  have the same dimensions, since we restrict ourselves to the full-order case. The closed-loop system, (21) and (23), is described by the state-space

equations

$$\begin{aligned} \begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{x}}_k \end{bmatrix} &= \mathbf{A}_{cl}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) \begin{bmatrix} \mathbf{x} \\ \mathbf{x}_k \end{bmatrix} + \mathbf{B}_{cl}(\boldsymbol{\theta}) \mathbf{w} \\ \mathbf{z} &= \mathbf{C}_{cl}(\boldsymbol{\theta}) \begin{bmatrix} \mathbf{x} \\ \mathbf{x}_k \end{bmatrix} + \mathbf{D}_{cl}(\boldsymbol{\theta}) \mathbf{w} \end{aligned} \quad (25)$$

where

$$\begin{aligned} \mathbf{A}_{cl}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) &= \begin{bmatrix} \mathbf{A}(\boldsymbol{\theta}) + \mathbf{B}_2 \mathbf{D}_k(\boldsymbol{\theta}) \mathbf{C}_2 & \mathbf{B}_2 \mathbf{C}_k(\boldsymbol{\theta}) \\ \mathbf{B}_k(\boldsymbol{\theta}) \mathbf{C}_2 & \mathbf{A}_k(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) \end{bmatrix} \\ \mathbf{B}_{cl}(\boldsymbol{\theta}) &= \begin{bmatrix} \mathbf{B}_1(\boldsymbol{\theta}) + \mathbf{B}_2 \mathbf{D}_k(\boldsymbol{\theta}) \mathbf{D}_{21} \\ \mathbf{B}_k(\boldsymbol{\theta}) \mathbf{D}_{21} \end{bmatrix} \\ \mathbf{C}_{cl}(\boldsymbol{\theta}) &= [\mathbf{C}_1(\boldsymbol{\theta}) + \mathbf{D}_{12} \mathbf{D}_k(\boldsymbol{\theta}) \mathbf{C}_2 \quad \mathbf{D}_{12} \mathbf{C}_k(\boldsymbol{\theta})] \\ \mathbf{D}_{cl}(\boldsymbol{\theta}) &= \mathbf{D}_{11}(\boldsymbol{\theta}) + \mathbf{D}_{12} \mathbf{D}_k(\boldsymbol{\theta}) \mathbf{D}_{21} \end{aligned} \quad (26)$$

Based on the parameter-dependent Lyapunov functions,  $V(\mathbf{x}, \boldsymbol{\theta}) = \mathbf{x}^T \mathbf{P}(\boldsymbol{\theta}) \mathbf{x}$ , there is an LPV controller  $\mathbf{K}(\boldsymbol{\theta})$  of the form of (23) that stabilizes the closed-loop system, (21) and (23), if and only if there exists  $\mathbf{P}(\boldsymbol{\theta}) = \mathbf{P}^T(\boldsymbol{\theta})$  such that (Gahinet et al. 1996)

$$\mathbf{P}(\boldsymbol{\theta}) > 0, \quad \frac{d}{dt}(\mathbf{x}^T \mathbf{P}(\boldsymbol{\theta}) \mathbf{x}) + \mathbf{z}^T \mathbf{z} - \gamma^2 \mathbf{w}^T \mathbf{w} < 0 \quad (27)$$

along all possible parameter trajectories,  $\forall(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) \in \boldsymbol{\Theta} \times \boldsymbol{\Phi}$ . Note that, unlike the single quadratic Lyapunov function case (Apkarian et al. 1995),  $\mathbf{P}(\boldsymbol{\theta})$ ,  $\mathbf{A}_k(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}})$ ,  $\mathbf{B}_k(\boldsymbol{\theta})$ ,  $\dots$ ,  $\mathbf{D}_k(\boldsymbol{\theta})$  and  $\mathbf{A}_{cl}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}})$ ,  $\mathbf{B}_{cl}(\boldsymbol{\theta})$ ,  $\dots$ ,  $\mathbf{D}_{cl}(\boldsymbol{\theta})$  do not depend affinely on the scheduled parameters  $\boldsymbol{\theta}$ . The inequality (27) leads to (Gahinet et al. 1996)

$$\begin{pmatrix} \mathbf{A}_{cl}^T(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) \mathbf{P}(\boldsymbol{\theta}) + \mathbf{P}(\boldsymbol{\theta}) \mathbf{A}_{cl}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) + \dot{\mathbf{P}}(\boldsymbol{\theta}) & \mathbf{P}(\boldsymbol{\theta}) \mathbf{B}_{cl}(\boldsymbol{\theta}) & \mathbf{C}_{cl}^T(\boldsymbol{\theta}) \\ \mathbf{B}_{cl}^T(\boldsymbol{\theta}) \mathbf{P}(\boldsymbol{\theta}) & -\gamma^2 \mathbf{I} & \mathbf{D}_{cl}^T(\boldsymbol{\theta}) \\ \mathbf{C}_{cl}(\boldsymbol{\theta}) & \mathbf{D}_{cl}(\boldsymbol{\theta}) & -\gamma^2 \mathbf{I} \end{pmatrix} < 0 \quad (28)$$

Introducing intermediate controller variables, i.e.  $\hat{\mathbf{A}}_k(\boldsymbol{\theta})$ ,  $\hat{\mathbf{B}}_k(\boldsymbol{\theta})$ ,  $\hat{\mathbf{C}}_k(\boldsymbol{\theta})$ , as (Apkarian and Adams 1998, Gahinet 1996)

$$\begin{aligned} \mathbf{A}_k(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) &= \mathbf{N}^{-1}(\boldsymbol{\theta}) \left( \mathbf{X}(\boldsymbol{\theta}) \dot{\mathbf{Y}}(\boldsymbol{\theta}) + \mathbf{N}(\boldsymbol{\theta}) \dot{\mathbf{M}}^T(\boldsymbol{\theta}) + \hat{\mathbf{A}}_k(\boldsymbol{\theta}) - \mathbf{X}(\boldsymbol{\theta}) (\mathbf{A}(\boldsymbol{\theta}) - \mathbf{B}_2 \mathbf{D}_k(\boldsymbol{\theta}) \mathbf{C}_2) \mathbf{Y}(\boldsymbol{\theta}) \right. \\ &\quad \left. - \hat{\mathbf{B}}_k(\boldsymbol{\theta}) \mathbf{C}_2 \mathbf{Y}(\boldsymbol{\theta}) - \mathbf{X}(\boldsymbol{\theta}) \mathbf{B}_2 \hat{\mathbf{C}}_k(\boldsymbol{\theta}) \right) \mathbf{M}^{-T}(\boldsymbol{\theta}) \end{aligned} \quad (29)$$

$$\mathbf{B}_k(\boldsymbol{\theta}) = \mathbf{N}^{-1}(\boldsymbol{\theta}) \left( \hat{\mathbf{B}}_k(\boldsymbol{\theta}) - \mathbf{X}(\boldsymbol{\theta}) \mathbf{B}_2 \mathbf{D}_k(\boldsymbol{\theta}) \right) \quad (30)$$

$$\mathbf{C}_k(\boldsymbol{\theta}) = \left( \hat{\mathbf{C}}_k(\boldsymbol{\theta}) - \mathbf{D}_k(\boldsymbol{\theta}) \mathbf{C}_2 \mathbf{Y}(\boldsymbol{\theta}) \right) \mathbf{M}^{-T}(\boldsymbol{\theta}) \quad (31)$$

where  $\mathbf{N}(\boldsymbol{\theta}) = -\mathbf{X}(\boldsymbol{\theta}) + \mathbf{Y}^{-1}(\boldsymbol{\theta})$ ,  $\dot{\mathbf{N}}(\boldsymbol{\theta}) = -\dot{\mathbf{X}}(\boldsymbol{\theta}) - \mathbf{Y}^{-1}(\boldsymbol{\theta}) \dot{\mathbf{Y}}(\boldsymbol{\theta}) \mathbf{Y}^{-1}(\boldsymbol{\theta})$ ,  $\mathbf{M}(\boldsymbol{\theta}) = \mathbf{Y}(\boldsymbol{\theta})$  and  $\dot{\mathbf{M}}(\boldsymbol{\theta}) = \dot{\mathbf{Y}}(\boldsymbol{\theta})$ . A pair of positive definite symmetric matrices  $(\mathbf{X}(\boldsymbol{\theta}), \mathbf{Y}(\boldsymbol{\theta}))$  is taken from the structure of the parameter-dependent Lyapunov variable,  $\mathbf{P}(\boldsymbol{\theta})$ , which is defined as (Wang and



Balakrishnan 2002)

$$\begin{aligned} \mathbf{P}(\boldsymbol{\theta}) &= \begin{bmatrix} \mathbf{X}(\boldsymbol{\theta}) & -(\mathbf{X}(\boldsymbol{\theta}) - \mathbf{Y}^{-1}(\boldsymbol{\theta})) \\ -(\mathbf{X}(\boldsymbol{\theta}) - \mathbf{Y}^{-1}(\boldsymbol{\theta})) & \mathbf{X}(\boldsymbol{\theta}) - \mathbf{Y}^{-1}(\boldsymbol{\theta}) \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{I}_p & \mathbf{X}(\boldsymbol{\theta}) \\ \mathbf{0}_{p \times p} & -(\mathbf{X}(\boldsymbol{\theta}) - \mathbf{Y}^{-1}(\boldsymbol{\theta})) \end{bmatrix} \begin{bmatrix} \mathbf{Y}(\boldsymbol{\theta}) & \mathbf{I}_p \\ \mathbf{Y}(\boldsymbol{\theta}) & \mathbf{0}_{p \times p} \end{bmatrix}^{-1} \end{aligned} \quad (32)$$

$$\dot{\mathbf{P}}(\boldsymbol{\theta}) = \begin{bmatrix} \dot{\mathbf{X}}(\boldsymbol{\theta}) & -\dot{\mathbf{X}}(\boldsymbol{\theta}) - \mathbf{Y}^{-1}(\boldsymbol{\theta})\dot{\mathbf{Y}}(\boldsymbol{\theta})\mathbf{Y}^{-1}(\boldsymbol{\theta}) \\ -\dot{\mathbf{X}}(\boldsymbol{\theta}) - \mathbf{Y}^{-1}(\boldsymbol{\theta})\dot{\mathbf{Y}}(\boldsymbol{\theta})\mathbf{Y}^{-1}(\boldsymbol{\theta}) & \dot{\mathbf{X}}(\boldsymbol{\theta}) + \mathbf{Y}^{-1}(\boldsymbol{\theta})\dot{\mathbf{Y}}(\boldsymbol{\theta})\mathbf{Y}^{-1}(\boldsymbol{\theta}) \end{bmatrix} \quad (33)$$

$$\begin{aligned} \mathbf{P}^{-1}(\boldsymbol{\theta}) &= \begin{bmatrix} \mathbf{Y}(\boldsymbol{\theta}) & \mathbf{Y}(\boldsymbol{\theta}) \\ \mathbf{Y}(\boldsymbol{\theta}) & (\mathbf{X}(\boldsymbol{\theta}) - \mathbf{Y}^{-1}(\boldsymbol{\theta}))^{-1}\mathbf{X}(\boldsymbol{\theta})\mathbf{Y}(\boldsymbol{\theta}) \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{Y}(\boldsymbol{\theta}) & \mathbf{I}_p \\ \mathbf{Y}(\boldsymbol{\theta}) & \mathbf{0}_{p \times p} \end{bmatrix} \begin{bmatrix} \mathbf{I}_p & \mathbf{X}(\boldsymbol{\theta}) \\ \mathbf{0}_{p \times p} & -(\mathbf{X}(\boldsymbol{\theta}) - \mathbf{Y}^{-1}(\boldsymbol{\theta})) \end{bmatrix}^{-1} \end{aligned} \quad (34)$$

where the positive definite symmetric matrices  $(\mathbf{X}(\boldsymbol{\theta}), \mathbf{Y}(\boldsymbol{\theta})) \in \mathbb{R}^{p \times p}$ ,  $\mathbf{X}(\boldsymbol{\theta}) - \mathbf{Y}^{-1}(\boldsymbol{\theta}) \geq 0$ , and  $\text{Rank}(\mathbf{X}(\boldsymbol{\theta}) - \mathbf{Y}^{-1}(\boldsymbol{\theta})) \leq p$  (Packard et al. 1991). Note that, the equations (29)–(31) show that  $\mathbf{A}_k(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}})$ ,  $\mathbf{B}_k(\boldsymbol{\theta})$  and  $\mathbf{C}_k(\boldsymbol{\theta})$  can not depend affinely on the scheduled parameters  $\boldsymbol{\theta}$  when the symmetric matrix  $\mathbf{X}$  or  $\mathbf{Y}$  is parameter-dependent. In this paper, we propose the intermediate controller variables, i.e.  $\hat{\mathbf{A}}_k(\boldsymbol{\theta})$ ,  $\hat{\mathbf{B}}_k(\boldsymbol{\theta})$ ,  $\hat{\mathbf{C}}_k(\boldsymbol{\theta})$  and  $\mathbf{D}_k(\boldsymbol{\theta})$ , and  $(\mathbf{X}(\boldsymbol{\theta}), \mathbf{Y}(\boldsymbol{\theta}))$  to depend affinely on the parameters  $\boldsymbol{\theta}$  as

$$\hat{\mathbf{A}}_k(\boldsymbol{\theta}) = \hat{\mathbf{A}}_{k_0} + \theta_1 \hat{\mathbf{A}}_{k_1} + \cdots + \theta_n \hat{\mathbf{A}}_{k_n} = \alpha_1 \tilde{\mathbf{A}}_{k_1} + \alpha_2 \tilde{\mathbf{A}}_{k_2} + \cdots + \alpha_m \tilde{\mathbf{A}}_{k_m} \quad (35)$$

$$\hat{\mathbf{B}}_k(\boldsymbol{\theta}) = \hat{\mathbf{B}}_{k_0} + \theta_1 \hat{\mathbf{B}}_{k_1} + \cdots + \theta_n \hat{\mathbf{B}}_{k_n} = \alpha_1 \tilde{\mathbf{B}}_{k_1} + \alpha_2 \tilde{\mathbf{B}}_{k_2} + \cdots + \alpha_m \tilde{\mathbf{B}}_{k_m} \quad (36)$$

$$\hat{\mathbf{C}}_k(\boldsymbol{\theta}) = \hat{\mathbf{C}}_{k_0} + \theta_1 \hat{\mathbf{C}}_{k_1} + \cdots + \theta_n \hat{\mathbf{C}}_{k_n} = \alpha_1 \tilde{\mathbf{C}}_{k_1} + \alpha_2 \tilde{\mathbf{C}}_{k_2} + \cdots + \alpha_m \tilde{\mathbf{C}}_{k_m} \quad (37)$$

$$\mathbf{D}_k(\boldsymbol{\theta}) = \mathbf{D}_{k_0} + \theta_1 \mathbf{D}_{k_1} + \cdots + \theta_n \mathbf{D}_{k_n} = \alpha_1 \tilde{\mathbf{D}}_{k_1} + \alpha_2 \tilde{\mathbf{D}}_{k_2} + \cdots + \alpha_m \tilde{\mathbf{D}}_{k_m} \quad (38)$$

$$\mathbf{X}(\boldsymbol{\theta}) = \mathbf{X}_0 + \theta_1 \mathbf{X}_1 + \cdots + \theta_n \mathbf{X}_n = \alpha_1 \hat{\mathbf{X}}_1 + \alpha_2 \hat{\mathbf{X}}_2 + \cdots + \alpha_m \hat{\mathbf{X}}_m \quad (39)$$

$$\mathbf{Y}(\boldsymbol{\theta}) = \mathbf{Y}_0 + \theta_1 \mathbf{Y}_1 + \cdots + \theta_n \mathbf{Y}_n = \alpha_1 \hat{\mathbf{Y}}_1 + \alpha_2 \hat{\mathbf{Y}}_2 + \cdots + \alpha_m \hat{\mathbf{Y}}_m \quad (40)$$

$$\dot{\mathbf{X}}(\boldsymbol{\theta}) = \dot{\theta}_1 \mathbf{X}_1 + \dot{\theta}_2 \mathbf{X}_2 + \cdots + \dot{\theta}_n \mathbf{X}_n = \beta_1 \tilde{\mathbf{X}}_1 + \beta_2 \tilde{\mathbf{X}}_2 + \cdots + \beta_m \tilde{\mathbf{X}}_m \quad (41)$$

$$\dot{\mathbf{Y}}(\boldsymbol{\theta}) = \dot{\theta}_1 \mathbf{Y}_1 + \dot{\theta}_2 \mathbf{Y}_2 + \cdots + \dot{\theta}_n \mathbf{Y}_n = \beta_1 \tilde{\mathbf{Y}}_1 + \beta_2 \tilde{\mathbf{Y}}_2 + \cdots + \beta_m \tilde{\mathbf{Y}}_m \quad (42)$$

Note that  $\tilde{\mathbf{X}}_j$  and  $\tilde{\mathbf{Y}}_j$ ,  $j = 1, \dots, m$ , map to  $\mathbf{X}_i$  and  $\mathbf{Y}_i$ ,  $i = 1, \dots, n$ , respectively in a similar manner to (12). This proposed technique offers obvious advantages in reducing computational burden and ease of controller implementation because the intermediate controller variables can be constructed as an affine matrix-valued function in the polytopic coordinates of the scheduled parameters. An existing method for computing the intermediate controller variables, that is based on explicit controller formulas (Gahinet 1996), is given in Appendix A. Define

$$\mathbf{P}_1(\boldsymbol{\theta}) = \begin{bmatrix} \mathbf{Y}(\boldsymbol{\theta}) & \mathbf{I}_p \\ \mathbf{Y}(\boldsymbol{\theta}) & \mathbf{0}_{p \times p} \end{bmatrix} \quad (43)$$

Following Apkarian and Adams (1998), by premultiplying the first row and postmultiplying the first column of (28) by  $\mathbf{P}_1^T(\boldsymbol{\theta})$  and  $\mathbf{P}_1(\boldsymbol{\theta})$  respectively and substituting (26) and (29)–(33) in

(28), we get

$$\begin{pmatrix} \dot{\mathbf{X}}(\boldsymbol{\theta}) + \left( \mathbf{X}(\boldsymbol{\theta})\mathbf{A}(\boldsymbol{\theta}) + \hat{\mathbf{B}}_k(\boldsymbol{\theta})\mathbf{C}_2 + (\star) \right) & \star \\ \hat{\mathbf{A}}_k^T(\boldsymbol{\theta}) + \mathbf{A}(\boldsymbol{\theta}) + \mathbf{B}_2\mathbf{D}_k(\boldsymbol{\theta})\mathbf{C}_2 & -\dot{\mathbf{Y}}(\boldsymbol{\theta}) + \left( \mathbf{A}(\boldsymbol{\theta})\mathbf{Y}(\boldsymbol{\theta}) + \mathbf{B}_2\hat{\mathbf{C}}_k(\boldsymbol{\theta}) + (\star) \right) \\ \mathbf{B}_1^T(\boldsymbol{\theta})\mathbf{X}(\boldsymbol{\theta}) + \mathbf{D}_{21}^T\hat{\mathbf{B}}_k^T(\boldsymbol{\theta}) & \mathbf{B}_1^T(\boldsymbol{\theta}) + \mathbf{D}_{21}^T\mathbf{D}_k^T(\boldsymbol{\theta})\mathbf{B}_2^T \\ \mathbf{C}_1(\boldsymbol{\theta}) + \mathbf{D}_{12}\mathbf{D}_k(\boldsymbol{\theta})\mathbf{C}_2 & \mathbf{C}_1(\boldsymbol{\theta})\mathbf{Y}(\boldsymbol{\theta}) + \mathbf{D}_{12}\hat{\mathbf{C}}_k(\boldsymbol{\theta}) \\ \star & \star \\ \star & \star \\ -\gamma\mathbf{I} & \star \\ \mathbf{D}_{11}(\boldsymbol{\theta}) + \mathbf{D}_{12}\mathbf{D}_k(\boldsymbol{\theta})\mathbf{D}_{21} & -\gamma\mathbf{I} \end{pmatrix} < 0 \quad (44)$$

where the notation  $\star$  represents a symmetric matrix block. Moreover, substituting (22) and (35)–(42) in (44), we have

$$\begin{aligned} & \sum_{i=1}^m \sum_{k=1}^m \alpha_i^2 \beta_k \begin{pmatrix} \tilde{\mathbf{X}}_k + \left( \hat{\mathbf{X}}_i \hat{\mathbf{A}}_i + \tilde{\mathbf{B}}_{k_i} \mathbf{C}_2 + (\star) \right) & \star \\ \tilde{\mathbf{A}}_{k_i}^T + \hat{\mathbf{A}}_i + \mathbf{B}_2 \tilde{\mathbf{D}}_{k_i} \mathbf{C}_2 & -\tilde{\mathbf{Y}}_k + \left( \hat{\mathbf{A}}_i \hat{\mathbf{Y}}_i + \mathbf{B}_2 \tilde{\mathbf{C}}_{k_i} + (\star) \right) \\ \hat{\mathbf{B}}_{1_i}^T \hat{\mathbf{X}}_i + \mathbf{D}_{21}^T \tilde{\mathbf{B}}_{k_i}^T & \hat{\mathbf{B}}_{1_i}^T + \mathbf{D}_{21}^T \tilde{\mathbf{D}}_{k_i}^T \mathbf{B}_2^T \\ \hat{\mathbf{C}}_{1_i} + \mathbf{D}_{12} \tilde{\mathbf{D}}_{k_i} \mathbf{C}_2 & \hat{\mathbf{C}}_{1_i} \hat{\mathbf{Y}}_i + \mathbf{D}_{12} \tilde{\mathbf{C}}_{k_i} \\ \star & \star \\ \star & \star \\ -\gamma\mathbf{I} & \star \\ \hat{\mathbf{D}}_{11_i} + \mathbf{D}_{12} \tilde{\mathbf{D}}_{k_i} \mathbf{D}_{21} & -\gamma\mathbf{I} \end{pmatrix} \\ & + 2 \sum_{i=1}^{m-1} \sum_{j=i+1}^m \sum_{k=1}^m \alpha_i \alpha_j \beta_k \begin{pmatrix} \tilde{\mathbf{X}}_k + \frac{1}{2} \left( \hat{\mathbf{X}}_j \hat{\mathbf{A}}_i + \tilde{\mathbf{B}}_{k_j} \mathbf{C}_2 + \hat{\mathbf{X}}_i \hat{\mathbf{A}}_j + \tilde{\mathbf{B}}_{k_i} \mathbf{C}_2 + (\star) \right) & \star \\ \frac{1}{2} \left( \tilde{\mathbf{A}}_{k_j}^T + \hat{\mathbf{A}}_i + \mathbf{B}_2 \tilde{\mathbf{D}}_{k_j} \mathbf{C}_2 + \tilde{\mathbf{A}}_{k_i}^T + \hat{\mathbf{A}}_j + \mathbf{B}_2 \tilde{\mathbf{D}}_{k_i} \mathbf{C}_2 \right) & -\tilde{\mathbf{Y}}_k + \frac{1}{2} \left( \hat{\mathbf{A}}_i \hat{\mathbf{Y}}_j + \mathbf{B}_2 \tilde{\mathbf{C}}_{k_j} + \hat{\mathbf{A}}_j \hat{\mathbf{Y}}_i + \mathbf{B}_2 \tilde{\mathbf{C}}_{k_i} + (\star) \right) \\ \frac{1}{2} \left( \hat{\mathbf{B}}_{1_i}^T \hat{\mathbf{X}}_j + \mathbf{D}_{21}^T \tilde{\mathbf{B}}_{k_j}^T + \hat{\mathbf{B}}_{1_j}^T \hat{\mathbf{X}}_i + \mathbf{D}_{21}^T \tilde{\mathbf{B}}_{k_i}^T \right) & \frac{1}{2} \left( \hat{\mathbf{B}}_{1_i}^T + \mathbf{D}_{21}^T \tilde{\mathbf{D}}_{k_j}^T \mathbf{B}_2^T + \hat{\mathbf{B}}_{1_j}^T + \mathbf{D}_{21}^T \tilde{\mathbf{D}}_{k_i}^T \mathbf{B}_2^T \right) \\ \frac{1}{2} \left( \hat{\mathbf{C}}_{1_i} + \mathbf{D}_{12} \tilde{\mathbf{D}}_{k_j} \mathbf{C}_2 + \hat{\mathbf{C}}_{1_j} + \mathbf{D}_{12} \tilde{\mathbf{D}}_{k_i} \mathbf{C}_2 \right) & \frac{1}{2} \left( \hat{\mathbf{C}}_{1_i} \hat{\mathbf{Y}}_j + \mathbf{D}_{12} \tilde{\mathbf{C}}_{k_j} + \hat{\mathbf{C}}_{1_j} \hat{\mathbf{Y}}_i + \mathbf{D}_{12} \tilde{\mathbf{C}}_{k_i} \right) \\ \star & \star \\ -\tilde{\mathbf{Y}}_k + \frac{1}{2} \left( \hat{\mathbf{A}}_i \hat{\mathbf{Y}}_j + \mathbf{B}_2 \tilde{\mathbf{C}}_{k_j} + \hat{\mathbf{A}}_j \hat{\mathbf{Y}}_i + \mathbf{B}_2 \tilde{\mathbf{C}}_{k_i} + (\star) \right) & \star \\ \frac{1}{2} \left( \hat{\mathbf{B}}_{1_i}^T + \mathbf{D}_{21}^T \tilde{\mathbf{D}}_{k_j}^T \mathbf{B}_2^T + \hat{\mathbf{B}}_{1_j}^T + \mathbf{D}_{21}^T \tilde{\mathbf{D}}_{k_i}^T \mathbf{B}_2^T \right) & \star \\ \frac{1}{2} \left( \hat{\mathbf{C}}_{1_i} \hat{\mathbf{Y}}_j + \mathbf{D}_{12} \tilde{\mathbf{C}}_{k_j} + \hat{\mathbf{C}}_{1_j} \hat{\mathbf{Y}}_i + \mathbf{D}_{12} \tilde{\mathbf{C}}_{k_i} \right) & \star \\ \star & \star \\ \star & \star \\ -\gamma\mathbf{I} & \star \\ \frac{1}{2} \left( \hat{\mathbf{D}}_{11_i} + \mathbf{D}_{12} \tilde{\mathbf{D}}_{k_j} \mathbf{D}_{21} + \hat{\mathbf{D}}_{11_j} + \mathbf{D}_{12} \tilde{\mathbf{D}}_{k_i} \mathbf{D}_{21} \right) & -\gamma\mathbf{I} \end{pmatrix} < 0 \quad (45)$$

in which the inequality (45) can be also rewritten as

$$\sum_{i=1}^m \sum_{k=1}^m \alpha_i^2 \beta_k \left( \Psi_{\text{cl}_{ii}} + \mathcal{Q}^T \hat{\mathbf{K}}_i^T \mathcal{P} + \mathcal{P}^T \hat{\mathbf{K}}_i \mathcal{Q} \right) + 2 \sum_{i=1}^{m-1} \sum_{j=i+1}^m \sum_{k=1}^m \alpha_i \alpha_j \beta_k \left( \frac{1}{2} (\Psi_{\text{cl}_{ij}} + \mathcal{Q}^T \hat{\mathbf{K}}_i^T \mathcal{P} + \mathcal{P}^T \hat{\mathbf{K}}_i \mathcal{Q} + \Psi_{\text{cl}_{ji}} + \mathcal{Q}^T \hat{\mathbf{K}}_j^T \mathcal{P} + \mathcal{P}^T \hat{\mathbf{K}}_j \mathcal{Q}) \right) < 0 \quad (46)$$

where

$$\begin{aligned} \Psi_{\text{cl}_{ii}} &= \begin{pmatrix} \tilde{\mathbf{X}}_k + \hat{\mathbf{X}}_i \hat{\mathbf{A}}_i + (\star) & \star & \star & \star \\ \hat{\mathbf{A}}_i & -\tilde{\mathbf{Y}}_k + \hat{\mathbf{A}}_i \hat{\mathbf{Y}}_i + (\star) & \star & \star \\ \hat{\mathbf{B}}_{1_i}^T \hat{\mathbf{X}}_i & \hat{\mathbf{B}}_{1_i}^T & -\gamma \mathbf{I} & \star \\ \hat{\mathbf{C}}_{1_i} & \hat{\mathbf{C}}_{1_i} \hat{\mathbf{Y}}_i & \hat{\mathbf{D}}_{11_i} & -\gamma \mathbf{I} \end{pmatrix} \\ \Psi_{\text{cl}_{ij}} &= \begin{pmatrix} \tilde{\mathbf{X}}_k + \hat{\mathbf{X}}_j \hat{\mathbf{A}}_i + (\star) & \star & \star & \star \\ \hat{\mathbf{A}}_i & -\tilde{\mathbf{Y}}_k + \hat{\mathbf{A}}_i \hat{\mathbf{Y}}_j + (\star) & \star & \star \\ \hat{\mathbf{B}}_{1_i}^T \hat{\mathbf{X}}_j & \hat{\mathbf{B}}_{1_i}^T & -\gamma \mathbf{I} & \star \\ \hat{\mathbf{C}}_{1_i} & \hat{\mathbf{C}}_{1_i} \hat{\mathbf{Y}}_j & \hat{\mathbf{D}}_{11_i} & -\gamma \mathbf{I} \end{pmatrix} \\ \Psi_{\text{cl}_{ji}} &= \begin{pmatrix} \tilde{\mathbf{X}}_k + \hat{\mathbf{X}}_i \hat{\mathbf{A}}_j + (\star) & \star & \star & \star \\ \hat{\mathbf{A}}_j & -\tilde{\mathbf{Y}}_k + \hat{\mathbf{A}}_j \hat{\mathbf{Y}}_i + (\star) & \star & \star \\ \hat{\mathbf{B}}_{1_j}^T \hat{\mathbf{X}}_i & \hat{\mathbf{B}}_{1_j}^T & -\gamma \mathbf{I} & \star \\ \hat{\mathbf{C}}_{1_j} & \hat{\mathbf{C}}_{1_j} \hat{\mathbf{Y}}_i & \hat{\mathbf{D}}_{11_j} & -\gamma \mathbf{I} \end{pmatrix} \\ \mathcal{Q} &= [\mathcal{C}, \quad \mathcal{D}_{21}, \quad \mathbf{0}_{(p+q_2) \times q_1}], \quad \mathcal{P} = [\tilde{\mathbf{B}}^T, \quad \mathbf{0}_{(p+m_2) \times m_1}, \quad \mathcal{D}_{12}^T] \\ \hat{\mathbf{K}}_i &= \begin{pmatrix} \tilde{\mathbf{A}}_{k_i} & \tilde{\mathbf{B}}_{k_i} \\ \tilde{\mathbf{C}}_{k_i} & \tilde{\mathbf{D}}_{k_i} \end{pmatrix}, \quad \tilde{\mathbf{B}} = \begin{bmatrix} \mathbf{I}_p & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_2 \end{bmatrix} \\ \mathcal{C} &= \begin{bmatrix} \mathbf{0} & \mathbf{I}_p \\ \mathbf{C}_2 & \mathbf{0} \end{bmatrix}, \quad \mathcal{D}_{12} = [\mathbf{0} \quad \mathbf{D}_{12}], \quad \mathcal{D}_{21} = \begin{bmatrix} \mathbf{0} \\ \mathbf{D}_{21} \end{bmatrix} \end{aligned} \quad (47)$$

By Lemma 2.1 and knowing the matrix vertices  $(\hat{\mathbf{X}}_i, \hat{\mathbf{Y}}_i)$ ,  $i = 1, 2, \dots, m$ , the system matrix vertices  $\hat{\mathbf{K}}_i$  can be determined from (46), that is an LMI in  $\hat{\mathbf{K}}_i$ , at all vertices for which  $(\hat{\mathbf{K}}_1, \hat{\mathbf{K}}_2, \dots, \hat{\mathbf{K}}_m)$  have to satisfy all of  $m^2(m+1)/2$  LMIs. Furthermore, knowing  $\tilde{\mathbf{A}}_{k_i}, \dots, \tilde{\mathbf{D}}_{k_i}$ , the controller system matrices  $\mathbf{A}_k(\theta, \dot{\theta}), \dots, \mathbf{D}_k(\theta)$  can be computed on-line in real-time using (29)–(31) and (38) with instantaneous measurement values of  $\theta$  and  $\dot{\theta}$ , where the proposed intermediate controller variables  $\hat{\mathbf{A}}_k(\theta), \hat{\mathbf{B}}_k(\theta), \hat{\mathbf{C}}_k(\theta)$  and  $\mathbf{D}_k(\theta)$ , and  $(\mathbf{X}(\theta), \mathbf{Y}(\theta))$  depend affinely on the parameters  $\theta$  and they can be computed on-line in real-time using (35)–(40). Hence, the proposed method reduces computational burden and eases controller implementation compared to the explicit controller formulas (Apkarian and Adams 1998, Gahinet 1996).

However, usually, the parameter derivatives either are not available or are difficult to estimate during system operation (Apkarian and Adams 1998). To avoid using the measured value of  $\dot{\theta}$ , we can constrain either  $\mathbf{X}(\theta)$  or  $\mathbf{Y}(\theta)$  to depend affinely on  $\theta$ . This yields  $\dot{\mathbf{X}}(\theta)\mathbf{Y}(\theta) + \dot{\mathbf{N}}(\theta)\mathbf{M}^T(\theta) = -(\mathbf{X}(\theta)\dot{\mathbf{Y}}(\theta) + \mathbf{N}(\theta)\dot{\mathbf{M}}^T(\theta)) = 0$  (Apkarian and Adams 1998), hence equation (29) becomes

$$\begin{aligned} \mathbf{A}_k(\theta) &= \mathbf{N}^{-1}(\theta) \left( \hat{\mathbf{A}}_k(\theta) - \mathbf{X}(\theta)(\mathbf{A}(\theta) - \mathbf{B}_2 \mathbf{D}_k(\theta) \mathbf{C}_2) \mathbf{Y}(\theta) \right. \\ &\quad \left. - \hat{\mathbf{B}}_k(\theta) \mathbf{C}_2 \mathbf{Y}(\theta) - \mathbf{X}(\theta) \mathbf{B}_2 \hat{\mathbf{C}}_k(\theta) \right) \mathbf{M}^{-T}(\theta) \end{aligned} \quad (48)$$

**Lemma 3.1:** (Gahinet and Apkarian 1994, Projection lemma) Given an inequality problem of the form

$$\Psi + \mathbf{Q}^T \mathbf{K}^T \mathbf{P} + \mathbf{P}^T \mathbf{K} \mathbf{Q} < 0 \quad (49)$$

where  $\Psi \in \mathbb{R}^{m \times m}$  is a symmetric matrix,  $\mathbf{Q}$  and  $\mathbf{P}$  are matrices with column dimension  $m$ . Let  $\mathbf{Q}_\perp$  and  $\mathbf{P}_\perp$  be any matrices whose columns form bases of the null spaces of  $\mathbf{Q}$  and  $\mathbf{P}$  respectively; the above problem is solvable for a matrix  $\mathbf{K}$  of compatible dimensions if and only if

$$\mathbf{Q}_\perp^T \Psi \mathbf{Q}_\perp < 0, \quad \mathbf{P}_\perp^T \Psi \mathbf{P}_\perp < 0 \quad (50)$$

By Lemmas 2.1 and 3.1, the LMIs of (46) are solvable at all vertices for  $\hat{\mathbf{K}}_i$  if and only if there exist a pair of positive definite symmetric matrices  $(\mathbf{X}(\theta), \mathbf{Y}(\theta))$  that satisfy the following theorem.

**Theorem 3.2:** There exists an LPV controller  $\mathbf{K}(\theta)$  guaranteeing the closed-loop system, (21) and (23), quadratic  $H_\infty$  performance  $\gamma$  along all possible parameter trajectories,  $\forall(\theta, \dot{\theta}) \in \Theta \times \Phi$ , if and only if the following LMI conditions hold for some positive definite symmetric matrices  $(\mathbf{X}(\theta), \mathbf{Y}(\theta))$ , which further satisfy  $\text{Rank}(\mathbf{X}(\theta) - \mathbf{Y}^{-1}(\theta)) \leq p$ .

$$\begin{pmatrix} \mathbf{N}_X & 0 \\ 0 & \mathbf{I} \end{pmatrix}^T \begin{pmatrix} \hat{\mathbf{A}}_i^T \hat{\mathbf{X}}_i + \hat{\mathbf{X}}_i \hat{\mathbf{A}}_i + \tilde{\mathbf{X}}_k & \hat{\mathbf{X}}_i \hat{\mathbf{B}}_{1_i} & \hat{\mathbf{C}}_{1_i}^T \\ & \hat{\mathbf{B}}_{1_i}^T \hat{\mathbf{X}}_i & -\gamma \mathbf{I} \\ & \hat{\mathbf{C}}_{1_i} & \hat{\mathbf{D}}_{11_i} \end{pmatrix} \begin{pmatrix} \mathbf{N}_X & 0 \\ 0 & \mathbf{I} \end{pmatrix} < 0 \quad (51)$$

$$\begin{pmatrix} \mathbf{N}_Y & 0 \\ 0 & \mathbf{I} \end{pmatrix}^T \begin{pmatrix} \hat{\mathbf{A}}_i \hat{\mathbf{Y}}_i + \hat{\mathbf{Y}}_i \hat{\mathbf{A}}_i^T - \tilde{\mathbf{Y}}_k & \hat{\mathbf{Y}}_i \hat{\mathbf{C}}_{1_i}^T & \hat{\mathbf{B}}_{1_i} \\ & \hat{\mathbf{C}}_{1_i}^T \hat{\mathbf{Y}}_i & -\gamma \mathbf{I} \\ & \hat{\mathbf{B}}_{1_i}^T & \hat{\mathbf{D}}_{11_i} \end{pmatrix} \begin{pmatrix} \mathbf{N}_Y & 0 \\ 0 & \mathbf{I} \end{pmatrix} < 0 \quad (52)$$

$$\begin{pmatrix} \mathbf{N}_X & 0 \\ 0 & \mathbf{I} \end{pmatrix}^T \begin{pmatrix} \hat{\mathbf{A}}_i^T \hat{\mathbf{X}}_j + \hat{\mathbf{X}}_j \hat{\mathbf{A}}_i + \hat{\mathbf{A}}_j^T \hat{\mathbf{X}}_i + \hat{\mathbf{X}}_i \hat{\mathbf{A}}_j + 2\tilde{\mathbf{X}}_k & \hat{\mathbf{X}}_i \hat{\mathbf{B}}_{1_j} + \hat{\mathbf{X}}_j \hat{\mathbf{B}}_{1_i} & \hat{\mathbf{C}}_{1_i}^T + \hat{\mathbf{C}}_{1_j}^T \\ & \hat{\mathbf{B}}_{1_i}^T \hat{\mathbf{X}}_j + \hat{\mathbf{B}}_{1_j}^T \hat{\mathbf{X}}_i & -2\gamma \mathbf{I} \\ & \hat{\mathbf{C}}_{1_i} + \hat{\mathbf{C}}_{1_j} & \hat{\mathbf{D}}_{11_i} + \hat{\mathbf{D}}_{11_j} \end{pmatrix} \begin{pmatrix} \mathbf{N}_X & 0 \\ 0 & \mathbf{I} \end{pmatrix} < 0 \quad (53)$$

$$\begin{pmatrix} \mathbf{N}_Y & 0 \\ 0 & \mathbf{I} \end{pmatrix}^T \begin{pmatrix} \hat{\mathbf{A}}_i \hat{\mathbf{Y}}_j + \hat{\mathbf{Y}}_j \hat{\mathbf{A}}_i^T + \hat{\mathbf{A}}_j \hat{\mathbf{Y}}_i + \hat{\mathbf{Y}}_i \hat{\mathbf{A}}_j^T - 2\tilde{\mathbf{Y}}_k & \hat{\mathbf{Y}}_i \hat{\mathbf{C}}_{1_j}^T + \hat{\mathbf{Y}}_j \hat{\mathbf{C}}_{1_i}^T & \hat{\mathbf{B}}_{1_i} + \hat{\mathbf{B}}_{1_j} \\ & \hat{\mathbf{C}}_{1_i}^T \hat{\mathbf{Y}}_j + \hat{\mathbf{C}}_{1_j}^T \hat{\mathbf{Y}}_i & -2\gamma \mathbf{I} \\ & \hat{\mathbf{B}}_{1_i}^T + \hat{\mathbf{B}}_{1_j}^T & \hat{\mathbf{D}}_{11_i} + \hat{\mathbf{D}}_{11_j} \end{pmatrix} \begin{pmatrix} \mathbf{N}_Y & 0 \\ 0 & \mathbf{I} \end{pmatrix} < 0 \quad (54)$$

$$\begin{pmatrix} \mathbf{X}_i & \mathbf{I} \\ \mathbf{I} & \mathbf{Y}_i \end{pmatrix} > 0, \quad \text{for } i, k = 1, \dots, m \text{ and } 1 \leq i < j \leq m \quad (55)$$

Note that Theorem 3.2 provides a new approach where an alternative to the multi-convexity approach (Apkarian and Tuan 2000) is given in Appendix B. In addition,  $\mathbf{N}_X$  and  $\mathbf{N}_Y$  denote bases of the null spaces of  $[\mathbf{C}_2, \mathbf{D}_{21}]$  and  $[\mathbf{B}_2^T, \mathbf{D}_{12}^T]$ , respectively. The inequality (55) ensures  $\mathbf{X}(\theta), \mathbf{Y}(\theta) > 0$  and  $\mathbf{X}(\theta) - \mathbf{Y}(\theta)^{-1} \geq 0$ .

#### 4. Numerical example

To demonstrate the effectiveness of the proposed approach, we consider the example of Leith and Leithhead (1999). For this example it has been shown that for an LPV plant model derived from the Jacobian, a common approach, with an LPV controller synthesized using the method of Apkarian et al. (1995), the closed-loop system is only stable when the LPV controller is applied

to the LPV model for a response to a step change in demand from -3 units to 0 units. However, when the same LPV controller is applied to the original nonlinear plant, the nonlinear closed-loop system appears to be unstable (Leith and Leithead 1999, Chumalee and Whidborne 2008).

Moreover, Chumalee and Whidborne (2008) have identified that the closed-loop instability occurs because the mismatch uncertainty between the Jacobian-based LPV model and the original nonlinear model is in a region close to the right-half  $s$ -plane. Furthermore, Shin (2002), Chumalee and Whidborne (2008) have shown that the state transformation methods (Balas 2002, Shamma and Cloutier 1993) give an LPV model that more accurately represents the nonlinear plant than the Jacobian linearization method. Hence, the closed-loop instability between the state transformation-based LPV controller and the original nonlinear model does not occur (Chumalee and Whidborne 2008).

Consider the nonlinear plant example taken from Leith and Leithead (1999)

$$\begin{aligned}\dot{x}_1(t) &= -x_1(t) + r(t) \\ \dot{x}_2(t) &= x_1(t) - |x_2(t)|x_2(t) - 10 \\ y(t) &= x_2(t)\end{aligned}\tag{56}$$

where  $t \in \mathbb{R}$  is time, both  $x_1(t), x_2(t) \in \mathbb{R}$  are the state,  $r(t), y(t) \in \mathbb{R}$  are the control input and the measurement output, respectively. Employing the state transformation methods (Balas 2002, Shamma and Cloutier 1993) yields a state transformation-based LPV model, taken from Shin (2002), Chumalee and Whidborne (2008), as

$$\begin{aligned}\begin{bmatrix} \dot{n}_1 \\ \dot{n}_2 \end{bmatrix} &= \begin{bmatrix} -1 - 2\theta & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \\ y &= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}\end{aligned}\tag{57}$$

where  $n_1 = x_1 - |x_2|x_2 - 10$ ,  $n_2 = x_2$ ,  $u = r - |x_2|x_2 - 10$ , and  $\theta = |n_2|$ . An LPV controller requirement is to ensure a step response settling time of less than 2 seconds with zero steady-state error (Leith and Leithead 1999). Thus, an LPV controller is synthesized with the criterion  $\|[W_1 S, W_2 K S]^T\| < 1$  where  $S = [I + GK]^{-1}$  and  $K S = K[I + GK]^{-1}$  are the sensitivity function and the control sensitivity function, respectively. The performance weighting,  $W_1$ , and robustness weighting,  $W_2$ , taken from Leith and Leithead (1999) are

$$\begin{aligned}W_1(s) &= \frac{0.5}{s + 0.002} \\ W_2(s) &= \frac{0.02s}{s + 1000}\end{aligned}\tag{58}$$

The objective of this mixed-sensitivity function is to shape the sensitivity function  $S$  and control sensitivity function  $K S$  with performance weighting functions  $W_1$  and robustness weighting functions  $W_2$  respectively.  $W_1$  has a low frequency gain 250 (48 dB), a high frequency gain 0, and a -3 dB frequency around 0.0024 rad/sec, which corresponds to 0.4% tracking error, infinity sensitivity function peak, and 0.0024 rad/sec desired bandwidth of sensitivity function, respectively.  $W_2$  has a low frequency gain 0, a high frequency gain 0.02 (-34 dB), and a -3 dB frequency around 676 rad/sec, which corresponds to infinity control sensitivity function peak, penalize limits on the response of the control signals, and 676 rad/sec desired bandwidth of control sensitivity function, respectively. Hence, we should get a controller that is good at command following, good at disturbance attenuation, low sensitivity to measurement noise, with

Table 1. Numerical comparisons of LPV synthesis techniques; an  $(\mathbf{X}(\boldsymbol{\theta}), \mathbf{Y}(\boldsymbol{\theta}))$  case

Condition $(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) \in [0, 10] \times [-10^{-6}, 10^{-6}]$	# of LMI	# of decision variables	CPU time (s)	performance $\gamma$
Multi-convexity <sup>a</sup>	16	46	0.5914	0.0909
Theorem 3.2	14	41	0.2076	0.0787
SQLF <sup>b</sup>	5	21	0.0289	0.0886
Condition $(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) \in [0, 10] \times [-25, 25]$	# of LMI	# of decision variables	CPU time (s)	performance $\gamma$
Multi-convexity <sup>a</sup>	16	46	0.5672	0.0935
Theorem 3.2	14	41	0.2268	0.0810
SQLF <sup>b</sup>	5	21	0.0289	0.0886
Condition $(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) \in [0, 10] \times [-10^6, 10^6]$	# of LMI	# of decision variables	CPU time (s)	performance $\gamma$
Multi-convexity <sup>a</sup>	16	46	0.7097	0.0974
Theorem 3.2	14	41	0.2515	0.0886
SQLF <sup>b</sup>	5	21	0.0289	0.0886
Condition $(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) \in [0, 10^4] \times [-25, 25]$	# of LMI	# of decision variables	CPU time (s)	performance $\gamma$
Multi-convexity <sup>a</sup>	16	46	0.9515	1.3982
Theorem 3.2	14	41	0.3067	0.7272
SQLF <sup>b</sup>	5	21	0.0604	1.6121

<sup>a</sup> Apkarian and Tuan (2000), shown in Appendix B.

<sup>b</sup> Single quadratic Lyapunov function (Apkarian et al. 1995).

reasonably small control efforts, and that is robustly stable to additive plant perturbations.

The results and numerical features of the LPV synthesis technique for the case where the pair  $(\mathbf{X}(\boldsymbol{\theta}), \mathbf{Y}(\boldsymbol{\theta}))$  are affine are presented in Table 1. It shows that, when the parameters have a large variation,  $(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) \in [0, 10^4] \times [-25, 25]$ , the computational time and the performance  $\gamma$  are increased for all three methodologies, i.e. multi-convexity (Apkarian and Tuan 2000), Theorem 3.2, and single quadratic Lyapunov function (SQLF) (Apkarian et al. 1995) approaches. The rate of parameters variation  $\dot{\boldsymbol{\theta}}$  does not affect the computational time and the performance  $\gamma$  for the SQLF (Apkarian et al. 1995) approach. This result from the fact that the rate of parameters variation is not used during computing LMIs. But, the rate of parameters variation affects the performance  $\gamma$  for both the multi-convexity and the proposed approaches as the performance  $\gamma$  is getting smaller when reducing the rate of parameters variation. The smallest performance  $\gamma$  can be achieved when the parameters are time-invariant or slowly varying,  $(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) \in [0, 10] \times [-10^{-6}, 10^{-6}]$ . The number of LMIs, decision variables, and the computational time are reduced while the achieved performance  $\gamma$  level is less conservative when using Theorem 3.2 compared with the multi-convexity technique, shown in Appendix B. The LMIs are solved using the MATLAB Robust Control Toolbox function (Balas et al. 2007), `mincx`, on a laptop computer (Intel(R) Core(TM)2 Duo Processor T7250 2.00 GHz with 2 GB DDR2 SDRAM).

A condition of  $(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) \in [0, 10] \times [-25, 25]$  and  $(\mathbf{X}, \mathbf{Y}(\boldsymbol{\theta}))$ ,  $\mathbf{X}$  is parameter-independent, are selected for synthesizing an LPV controller in order to demonstrate the transient response to a step change in demand from -3 units to 0 units of the closed-loop system between the LPV controller with the original nonlinear model (56) through nonlinear simulation in MATLAB Simulink environment. The performances  $\gamma$  of 0.0936, 0.0810, and 0.0886 are obtained via multi-convexity (Apkarian and Tuan 2000), Theorem 3.2, and single quadratic Lyapunov function (Apkarian et al. 1995) approaches, respectively.

Figure 1 shows the step responses of three LPV controllers. One can see from Figure 1 that the

Table 2. Computational time of  $\hat{\mathbf{A}}_k(\boldsymbol{\theta})$ ,  $\hat{\mathbf{B}}_k(\boldsymbol{\theta})$ ,  $\hat{\mathbf{C}}_k(\boldsymbol{\theta})$  and  $\mathbf{D}_k(\boldsymbol{\theta})$

Method	CPU time (us)
Explicit formulas <sup>a</sup>	1635
Proposed method <sup>b</sup>	284

<sup>a</sup> Gahinet (1996), Appendix A.

<sup>b</sup> Equations (35)–(38).

transient responses of the controllers from multi-convexity (Apkarian and Tuan 2000) and single quadratic Lyapunov function (Apkarian et al. 1995) approaches are slightly over damped while the transient response of the approach-based controller is slightly under damped. The control inputs  $r(t)$  to these three transient responses are also shown in Figure 2. In addition, the derivative of the parameter-dependent Lyapunov functions  $\frac{d}{dt}(V(\mathbf{x}, \boldsymbol{\theta}))$  are shown in Figure 3 to illustrate stability of the closed loop system for all three methodologies used. The numerical simulation results presented in Figures 1–3 verifies that both single quadratic Lyapunov function (Apkarian et al. 1995) and multi-convexity technique (Apkarian and Tuan 2000) approaches are more conservative than the proposed approaches. Furthermore, Table 2 shows computational time of the intermediate controller variables using Explicit formulas (Gahinet 1996), shown in Appendix A, and the proposed technique, equations (35)–(38). One can see from Table 2 that our proposed technique has a lower computational time and ease controller implementation compared with the multi-convexity technique.

## 5. Conclusion

In this paper, new sufficient conditions for gain-scheduled  $H_\infty$  performance analysis and synthesis for a class of affine LPV systems using parameter-dependent Lyapunov functions are as proposed in Theorem 3.2. Compared with the multi-convexity technique (Apkarian and Tuan 2000), shown in Appendix B, fewer LMIs and decision variables are required and the computational time is lower while the achieved performance  $\gamma$  is improved. The analysis and synthesis conditions are represented in the form of a finite number of LMIs. In contrast with the explicit controller formulas (Apkarian and Adams 1998, Gahinet 1996), shown in Appendix A, the intermediate controller variables, i.e.  $\hat{\mathbf{A}}_k(\boldsymbol{\theta})$ ,  $\hat{\mathbf{B}}_k(\boldsymbol{\theta})$ ,  $\hat{\mathbf{C}}_k(\boldsymbol{\theta})$  and  $\mathbf{D}_k(\boldsymbol{\theta})$ , are proposed to be constructed as an affine matrix-valued function in the polytopic coordinates of the scheduled parameters without the need for constraints on the  $\mathbf{D}_{12}$  and  $\mathbf{D}_{21}$  matrices. Hence, this reduces the computational burden and eases controller implementation.

The approach was then applied to synthesize an LPV controller where it was tested with the nonlinear model taken from Leith and Leithead (1999). The nonlinear simulation results show the effectiveness of the proposed approach.

## Appendix A. Explicit Controller Formulas

**Algorithm A.1:** (Apkarian and Adams 1998, Gahinet 1996) Computation of  $\hat{\mathbf{A}}_k(\boldsymbol{\theta})$ ,  $\hat{\mathbf{B}}_k(\boldsymbol{\theta})$ ,  $\hat{\mathbf{C}}_k(\boldsymbol{\theta})$ , and  $\mathbf{D}_k(\boldsymbol{\theta})$ .

Step 1: Set  $\mathbf{D}_k(\boldsymbol{\theta}) = (\mathbf{D}_{12}^+ \mathbf{D}_{12}) \mathbf{D}_0(\boldsymbol{\theta}) (\mathbf{D}_{21} \mathbf{D}_{21}^+)$ , where  $\mathbf{D}_0(\boldsymbol{\theta})$  is any matrix such that  $\sigma_{\max}(\mathbf{D}_{11}(\boldsymbol{\theta}) + \mathbf{D}_{12} \mathbf{D}_0(\boldsymbol{\theta}) \mathbf{D}_{21}) < \gamma$ . This amounts to solving a Parrott problem.

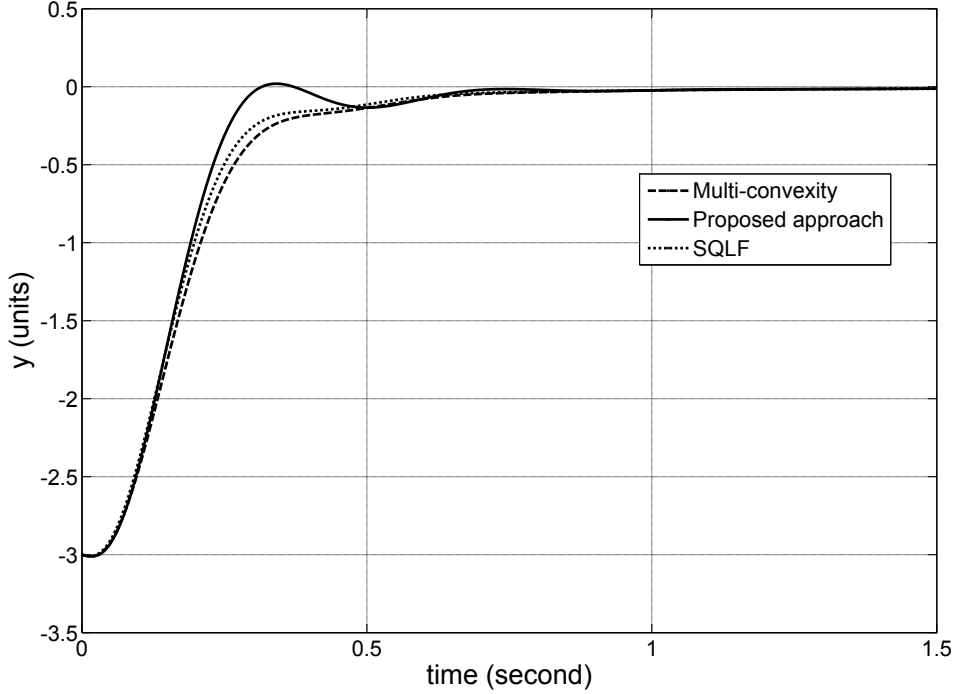


Figure 1. Nonlinear step response from -3 to 0 with the original nonlinear plant

Step 2: Compute the least-squares solutions of

$$\begin{pmatrix} \mathbf{0} & [\mathbf{D}_{21} \mathbf{0}] \\ [\mathbf{D}_{21}^T] & \begin{bmatrix} -\gamma \mathbf{I} & \mathbf{D}_{cl}^T(\theta) \\ \mathbf{D}_{cl}(\theta) & -\gamma \mathbf{I} \end{bmatrix} \end{pmatrix} \begin{pmatrix} \Theta_{B1}(\theta) \\ [\dagger] \end{pmatrix} = - \begin{pmatrix} \mathbf{C}_2 \\ \mathbf{B}_1^T(\theta) \mathbf{X}(\theta) \\ \mathbf{C}_1(\theta) + \mathbf{D}_{12} \mathbf{D}_k(\theta) \mathbf{C}_2 \end{pmatrix} \quad (\text{A1})$$

$$\begin{pmatrix} \mathbf{0} & [\mathbf{0} \mathbf{D}_{12}^T] \\ [\mathbf{0}] & \begin{bmatrix} -\gamma \mathbf{I} & \mathbf{D}_{cl}^T(\theta) \\ \mathbf{D}_{cl}(\theta) & -\gamma \mathbf{I} \end{bmatrix} \\ [\mathbf{D}_{12}] & \end{pmatrix} \begin{pmatrix} \Theta_{C1}(\theta) \\ [\dagger] \end{pmatrix} = - \begin{pmatrix} \mathbf{B}_2^T \\ [(\mathbf{B}_1(\theta) + \mathbf{B}_2 \mathbf{D}_k(\theta) \mathbf{D}_{21})^T] \\ \mathbf{C}_1(\theta) \mathbf{Y}(\theta) \end{pmatrix} \quad (\text{A2})$$

where  $\mathbf{D}_{cl}(\theta) = \mathbf{D}_{11}(\theta) + \mathbf{D}_{12} \mathbf{D}_k(\theta) \mathbf{D}_{21}$  and  $\dagger$  denotes matrices without interest here.

Step 3: If  $\Pi_{21} \mathbf{C}_2 = 0$ , set  $\Theta_{B2}(\theta) = 0$ . Otherwise, compute  $\Theta_{B2}(\theta)$  such that

$$\Psi + \mathbf{C}_2^T \Pi_{21} \Theta_{B2}(\theta) + \Theta_{B2}^T(\theta) \Pi_{21} \mathbf{C}_2 < 0 \quad (\text{A3})$$

where  $\Pi_{21} = \mathbf{I} - \mathbf{D}_{21} \mathbf{D}_{21}^+$  and

$$\begin{aligned} \Psi &= \mathbf{A}^T(\theta) \mathbf{X}(\theta) + \mathbf{X}(\theta) \mathbf{A}(\theta) + \dot{\mathbf{X}}(\theta) + \Theta_{B1}^T(\theta) \mathbf{C}_2 + \mathbf{C}_2^T \Theta_{B1}(\theta) \\ &+ \begin{bmatrix} \mathbf{B}_1^T(\theta) \mathbf{X}(\theta) + \mathbf{D}_{21}^T \Theta_{B1}(\theta) \\ \mathbf{C}_1(\theta) + \mathbf{D}_{12} \mathbf{D}_k(\theta) \mathbf{C}_2 \end{bmatrix}^T \begin{bmatrix} \gamma \mathbf{I} & -\mathbf{D}_{cl}^T(\theta) \\ -\mathbf{D}_{cl}(\theta) & \gamma \mathbf{I} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{B}_1^T(\theta) \mathbf{X}(\theta) + \mathbf{D}_{21}^T \Theta_{B1}(\theta) \\ \mathbf{C}_1(\theta) + \mathbf{D}_{12} \mathbf{D}_k(\theta) \mathbf{C}_2 \end{bmatrix} \end{aligned} \quad (\text{A4})$$

Similarly, set  $\Theta_{C2}(\theta) = 0$  if  $\Pi_{12} \mathbf{B}_2^T = 0$ . Otherwise, compute  $\Theta_{C2}(\theta)$  such that

$$\Pi + \mathbf{B}_2 \Pi_{12} \Theta_{C2}(\theta) + \Theta_{C2}^T(\theta) \Pi_{12} \mathbf{B}_2^T < 0 \quad (\text{A5})$$



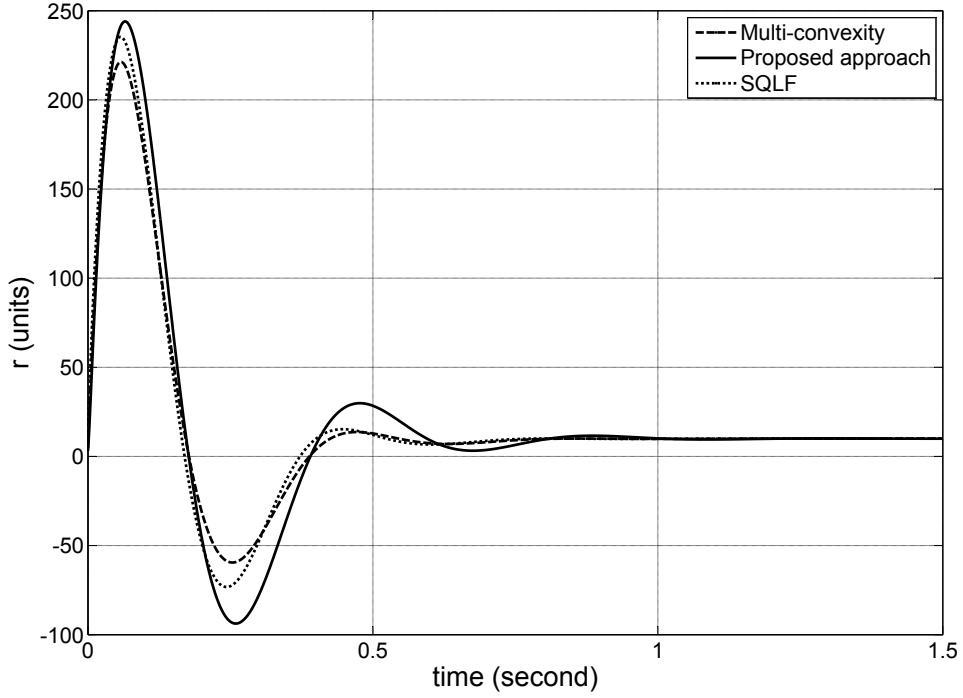


Figure 2. Control input to the original nonlinear plant

where  $\Pi_{12} = \mathbf{I} - \mathbf{D}_{12}^+ \mathbf{D}_{12}$  and

$$\begin{aligned} \Pi = & \mathbf{A}(\theta)\mathbf{Y}(\theta) + \mathbf{Y}(\theta)\mathbf{A}^T(\theta) - \dot{\mathbf{Y}}(\theta) + \mathbf{B}_2\boldsymbol{\Theta}_{C1}^T(\theta) + \boldsymbol{\Theta}_{C1}^T(\theta)\mathbf{B}_2^T \\ & + \left[ \begin{pmatrix} \mathbf{B}_1(\theta) + \mathbf{B}_2\mathbf{D}_k(\theta)\mathbf{D}_{21} \end{pmatrix}^T \right]^T \begin{bmatrix} \gamma\mathbf{I} & -\mathbf{D}_{cl}^T(\theta) \\ -\mathbf{D}_{cl}(\theta) & \gamma\mathbf{I} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{B}_1(\theta) + \mathbf{B}_2\mathbf{D}_k(\theta)\mathbf{D}_{21} \\ \mathbf{C}_1(\theta)\mathbf{Y}(\theta) + \mathbf{D}_{12}\boldsymbol{\Theta}_{C1}(\theta) \end{bmatrix}^T \end{aligned} \quad (\text{A6})$$

Step 4: Compute  $\hat{\mathbf{A}}_k(\theta)$ ,  $\hat{\mathbf{B}}_k(\theta)$ , and  $\hat{\mathbf{C}}_k(\theta)$  as

$$\begin{aligned} \hat{\mathbf{A}}_k(\theta) = & -(\mathbf{A}(\theta) + \mathbf{B}_2\mathbf{D}_k(\theta)\mathbf{C}_2)^T + \left[ (\mathbf{X}(\theta)\mathbf{B}_1(\theta) + \hat{\mathbf{B}}_k(\theta)\mathbf{D}_{21}) (\mathbf{C}_1(\theta) + \mathbf{D}_{12}\mathbf{D}_k(\theta)\mathbf{C}_2)^T \right] \\ & \times \begin{bmatrix} -\gamma\mathbf{I} & \mathbf{D}_{cl}^T(\theta) \\ \mathbf{D}_{cl}(\theta) & -\gamma\mathbf{I} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{B}_1(\theta) + \mathbf{B}_2\mathbf{D}_k(\theta)\mathbf{D}_{21} \\ \mathbf{C}_1(\theta)\mathbf{Y}(\theta) + \mathbf{D}_{12}\hat{\mathbf{C}}_k(\theta) \end{bmatrix}^T \end{aligned} \quad (\text{A7})$$

$$\hat{\mathbf{B}}_k(\theta) = (\boldsymbol{\Theta}_{B1}(\theta) + \Pi_{21}\boldsymbol{\Theta}_{B2}(\theta))^T \quad (\text{A8})$$

$$\hat{\mathbf{C}}_k(\theta) = \boldsymbol{\Theta}_{C1}(\theta) + \Pi_{12}\boldsymbol{\Theta}_{C2}(\theta) \quad (\text{A9})$$

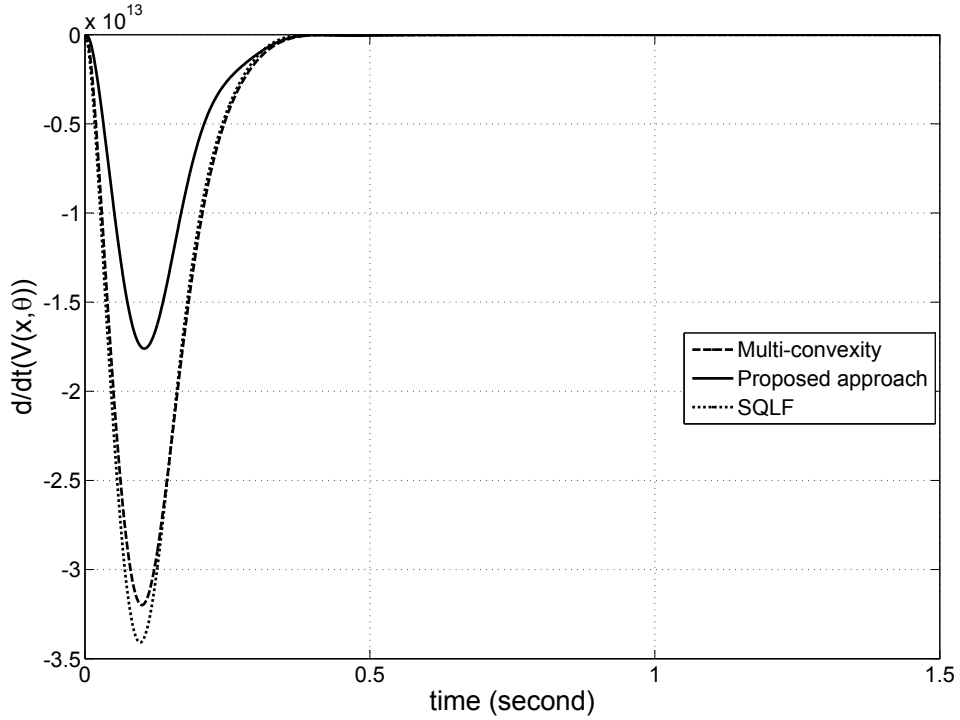


Figure 3. The derivative of the parameter-dependent Lyapunov functions  $\frac{d}{dt}(V(\mathbf{x}, \boldsymbol{\theta}))$

## Appendix B. Multi-Convexity Approach

A generalized affine LPV model with state-space realization taken from Apkarian and Tuan (2000) is

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}(\boldsymbol{\theta})\mathbf{x} + \mathbf{B}_1(\boldsymbol{\theta})\mathbf{w} + \mathbf{B}_2(\boldsymbol{\theta})\mathbf{u} \\ \mathbf{z} &= \mathbf{C}_1(\boldsymbol{\theta})\mathbf{x} + \mathbf{D}_{11}(\boldsymbol{\theta})\mathbf{w} + \mathbf{D}_{12}(\boldsymbol{\theta})\mathbf{u} \\ \mathbf{y} &= \mathbf{C}_2(\boldsymbol{\theta})\mathbf{x} + \mathbf{D}_{21}(\boldsymbol{\theta})\mathbf{w}\end{aligned}\tag{B1}$$

Note that, unlike the system (21), the state-space data  $\mathbf{B}_2(\cdot)$ ,  $\mathbf{C}_2(\cdot)$ ,  $\mathbf{D}_{12}(\cdot)$ , and  $\mathbf{D}_{21}(\cdot)$  of the system (B1) are parameter-dependent on  $\boldsymbol{\theta}$ .

**Theorem B.1:** (Apkarian and Tuan 2000) *For the affine LPV system (B1), there exists an LPV controller (23) solution to the LPV control problem with guaranteed  $L_2$ -gain performance with level  $\gamma$  along all possible parameter trajectories,  $\forall(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) \in \boldsymbol{\Theta} \times \boldsymbol{\Phi}$ , whenever there exist symmetric matrices  $\mathbf{X}_0, \mathbf{X}_1, \dots, \mathbf{X}_n$  and  $\mathbf{Y}_0, \mathbf{Y}_1, \dots, \mathbf{Y}_n$  and scalars  $\lambda_0, \lambda_1, \dots, \lambda_n, \mu_0, \mu_1$ ,*

...,  $\mu_n$  and  $\sigma$  such that

$$\begin{bmatrix} \tilde{\mathbf{X}}_k + \hat{\mathbf{X}}_j \hat{\mathbf{A}}_j + \hat{\mathbf{A}}_j^T \hat{\mathbf{X}}_j & \hat{\mathbf{X}}_j \hat{\mathbf{B}}_{1j} & \hat{\mathbf{C}}_{1j}^T \\ \hat{\mathbf{B}}_{1j}^T \hat{\mathbf{X}}_j & -\gamma \mathbf{I} & \hat{\mathbf{D}}_{11j}^T \\ \hat{\mathbf{C}}_{1j} & \hat{\mathbf{D}}_{11j} & -\gamma \mathbf{I} \end{bmatrix} - \sigma \begin{bmatrix} \hat{\mathbf{C}}_{2j}^T \\ \hat{\mathbf{D}}_{21j}^T \\ \mathbf{0} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{C}}_{2j} & \hat{\mathbf{D}}_{21j} & \mathbf{0} \end{bmatrix} < - \left( \lambda_0 + \sum_{i=1}^n \theta_i^2 \lambda_i \right) \mathbf{I} \quad (\text{B2})$$

$$\begin{bmatrix} -\tilde{\mathbf{Y}}_k + \hat{\mathbf{Y}}_j \hat{\mathbf{A}}_j^T + \hat{\mathbf{A}}_j \hat{\mathbf{Y}}_j & \hat{\mathbf{Y}}_j \hat{\mathbf{C}}_{1j}^T & \hat{\mathbf{B}}_{1j} \\ \hat{\mathbf{C}}_{1j}^T \hat{\mathbf{Y}}_j & -\gamma \mathbf{I} & \hat{\mathbf{D}}_{11j}^T \\ \hat{\mathbf{B}}_{1j}^T & \hat{\mathbf{D}}_{11j} & -\gamma \mathbf{I} \end{bmatrix} - \sigma \begin{bmatrix} \hat{\mathbf{B}}_{2j}^T \\ \hat{\mathbf{D}}_{12j}^T \\ \mathbf{0} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{B}}_{2j} & \hat{\mathbf{D}}_{12j} & \mathbf{0} \end{bmatrix} < - \left( \mu_0 + \sum_{i=1}^n \theta_i^2 \mu_i \right) \mathbf{I} \quad (\text{B3})$$

$$\begin{bmatrix} \hat{\mathbf{X}}_j & \mathbf{I} \\ \mathbf{I} & \hat{\mathbf{Y}}_j \end{bmatrix} > \mathbf{0} \quad (\text{B4})$$

$$\begin{bmatrix} \mathbf{X}_i \mathbf{A}_i + \mathbf{A}_i^T \mathbf{X}_i & \mathbf{X}_i \mathbf{B}_{1i} \\ \mathbf{B}_{1i}^T \mathbf{X}_i & \mathbf{0} \end{bmatrix} - \sigma \begin{bmatrix} \mathbf{C}_{2i}^T \mathbf{C}_{2i} & \mathbf{C}_{2i}^T \mathbf{D}_{21i} \\ \mathbf{D}_{21i}^T \mathbf{C}_{2i} & \mathbf{D}_{21i}^T \mathbf{D}_{21i} \end{bmatrix} \geq -\lambda_i \mathbf{I} \quad (\text{B5})$$

$$\begin{bmatrix} \mathbf{Y}_i \mathbf{A}_i^T + \mathbf{A}_i \mathbf{Y}_i & \mathbf{Y}_i \mathbf{C}_{1i}^T \\ \mathbf{C}_{1i} \mathbf{Y}_i & \mathbf{0} \end{bmatrix} - \sigma \begin{bmatrix} \mathbf{B}_{2i} \mathbf{B}_{2i}^T & \mathbf{B}_{2i} \mathbf{D}_{12i}^T \\ \mathbf{D}_{12i} \mathbf{B}_{2i}^T & \mathbf{D}_{12i} \mathbf{D}_{12i}^T \end{bmatrix} \geq -\mu_i \mathbf{I} \quad (\text{B6})$$

$$\lambda_0 \geq 0, \quad \lambda_i \geq 0, \quad \mu_0 \geq 0, \quad \mu_i \geq 0 \quad (\text{B7})$$

for  $j, k=1, 2, \dots, m$  and  $i=1, 2, \dots, n$

Note that  $\hat{\mathbf{X}}_j$  and  $\hat{\mathbf{Y}}_j$ ,  $j = 1, \dots, m$ , map to  $\mathbf{X}_i$  and  $\mathbf{Y}_i$ ,  $i = 0, \dots, n$ , respectively in a similar manner to (10). Having determined  $\mathbf{X}(\boldsymbol{\theta})$  and  $\mathbf{Y}(\boldsymbol{\theta})$  using Theorem B.1,  $\mathbf{N}(\boldsymbol{\theta})$  and  $\mathbf{M}(\boldsymbol{\theta})$  can be determined from the factorization problem:

$$\mathbf{I} - \mathbf{X}(\boldsymbol{\theta})\mathbf{Y}(\boldsymbol{\theta}) = \mathbf{N}(\boldsymbol{\theta})\mathbf{M}^T(\boldsymbol{\theta}) \quad (\text{B8})$$

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# Gain-scheduled $H^\infty$ control via parameter-dependent Lyapunov functions

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